Group Classification and Similarity Solutions of Klein Gordon Equations on a Sphere

By

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This thesis, written by **KHALID ALI AL-ANEZY** under the direction of his thesis advisor and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE IN MATHEMATICS.**

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This thesis is dedicated to my parents

For their endless love, support and encouragement

ACNOELEDGEMENT

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THESIS ABSTRACT

Name : Khaled Ali Ayed Al-Anezy

Title : Group Classification and Similarity Solutions of Klein Gordon Equations on a Sphere

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Lie symmetry method is a technique to find exact solutions of differential equations. One of the significant applications of Lie symmetry theory is to achieve a complete classification of Lie symmetries and symmetry reductions of differential equations. This project is concerned with carrying out a complete symmetry analysis of Klein Gordon Equations of the form

$$u_{tt} = \Delta u + f(u), \qquad (*)$$

on a sphere where Δ denotes the Laplacian operator on the sphere.

We will separately investigate the following two cases of the equation (*):

Case I: The function $f(u)$ is nonlinear.

Case II: The function $f(u)$ is linear (i.e. $f(u) = au$).

For the first case, our aim is to:

1. Find the minimal symmetry algebra.

2. Find all functions $f(u)$ which give larger symmetry algebras and determine these symmetry algebras.

3. Find some symmetry reductions and exact solutions for each case of $f(u)$.

And for the second case, our aim is to:

1. Find the symmetry algebra.

2. Find some symmetry reductions and exact solutions.

ملخص الرسالة

الاسم : خالد علي عايد العنزي

عنوان الرسالة : تصنيف الزمر وإيجاد حلول تناظرية لمعادلات كلاين غوردون على سطح كروي

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طريقة تناظر "لي" هي إحدى الطرق لإيجاد حلول دقيقة للمعادلات التفاضلية، وأحد أهم التطبيقات لنظرية تناظر "لي" هو الوصول للتصنيف الكامل لتناظرات "لي" والاختزالات والحلول التناظرية للمعادلات التفاضلية. هذا البحث معني بالتحليل التماثلي لمعادلات كلاين غوردون على الصورة

$$u_{tt} = \Delta u + f(u), \qquad (*)$$

وسوف يدرس مؤثر لابلاس Δ على سطح كروي، وسوف تدرس المعادلة (*) في حالتين مختلفتين:

الأولى: الدالة ***f(u)*** غير خطية

الثانية: الدالة ***f(u)*** خطية (على الصورة: ***f(u) =ua***)

CHAPTER 1
Introduction and problem formulation

The mathematical modeling of most of the natural and physical processes leads to nonlinear differential equations whose analytic solutions are hard to find. Therefore, investigations related to simplifications of nonlinear differential equations and construction of their exact solutions become significant in the analysis of nonlinear differential equations. Lie symmetry method has proven to be a powerful technique for analyzing non-linear ODEs and PDEs. It provides most widely applicable technique to find exact solutions of differential equations and contains, as particular case cf. [33], many efficient methods for solving differential equations like traveling wave solutions, self-similar solutions and exponential self-similar solutions. The classical Lie symmetry theory to study differential equations was developed by Sophus Lie more than a century ago. A modern treatment of the classical Lie symmetry theory was provided by Ovsiannikov [32]. Since the modern treatment by Ovsiannikov, the theory has substantially grown and found widespread uses. A large amount of literature about the classical Lie symmetry theory, its applications and its extensions is available, e.g. [2, 6, 7, 8, 11, 15, 17, 18, 19, 20, 21, 30, 31, 32, 36].

The Klein Gordon equation in curved spaces plays a significant role in study of the relativistic equations of motion. It is the equation of motion of a quantum scalar, a field whose quanta are spinless particles. The equation was named after the physicists Oskar Klein and Walter Gordon, who in 1926 proposed that it describes the relativistic quantum mechanical equation of the electrons.

The main aim of this work is to investigate the group classification problem which, in general, consists of two main steps. The first step is finding the Lie symmetries of a differential equation with arbitrary f. The second step is determining all possible forms of f for which larger symmetry groups exist. The first group classification problem was carried out by Ovsiannikov [32] who classified all forms of the non-linear heat equation

$$u_t = \left(f\left(u\right)u_x\right)_x .$$

Since then a number of articles on symmetry analysis and classification problem for non-linear PDEs have appeared in literature, cf. [3, 4, 5, 9, 10, 12, 14, 16, 22, 23, 24, 25, 26, 27, 28, 34, 35, 37, 38]. In the present work, we propose to perform the complete group classification of nonlinear Klein Gordon equations on a sphere.

Section 1.1 contains some basic ideas about differential geometry of surfaces, needed for setting up the problems of this research. The problem formulation is provided in section 1.2.

1.1 Some basic definitions from differential geometry

To formulate our problem clearly we need some basic background in differential geometry, [13], specially the definition of Laplacian on surfaces. In this section we define the metric or first fundamental form of surfaces.

Definition:

Let $X(x, y)$ be a coordinate patch or parameterization of a surface M. Then,

$$g = ds^2 = Edx^2 + 2Fdxdy + Gdy^2,$$

is called the first fundamental form or Riemannian metric of the surface, where

$$E = X_x \cdot X_x, \quad F = X_x \cdot X_y, \quad G = X_y \cdot X_y.$$

Setting

$$g_{11} = E = X_x \cdot X_x, \quad g_{12} = g_{21} = F = X_x \cdot X_y, \quad g_{22} = G = X_y \cdot X_y,$$

leads to the classical notation of the metric

$$g = ds^2 = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2,$$

or in the form of a symmetric matrix

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix}.$$

Note that

$$det\,(g) = g_{11}g_{22} - (g_{12})^2 = EG - F^2 = |X_x \times X_y| \neq 0,$$

therefore,

$$g^{-1} = \begin{bmatrix} g^{11} & g^{12} \\ g^{12} & g^{22} \end{bmatrix} = \frac{1}{\det(g)} \begin{bmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{bmatrix} = \left(g^{ij}\right).$$

The main operator involved in our investigation is Laplacian on surfaces which is defined below.

Definition:

Consider a surface with a metric g. Then the Laplacian on the surface is defined as

$$\triangle u = \frac{1}{\sqrt{|\det(g)|}} \frac{\partial}{\partial x^i} \left(\sqrt{|\det(g)|} g^{ij} \frac{\partial u}{\partial x^j} \right), \tag{1.1}$$

where the summation is taken over repeated indices.

1.2 Problem formulation of Klein Gordon equations

The objective of the study is to carry out complete symmetry group classification and symmetry analysis of Klein Gordon equations of the form

$$u_{tt} = \triangle u + f(u), \tag{1.2}$$

where $\triangle$ denotes the Laplacian operator on a sphere defined in the previous section.

Consider the sphere

$$x^2 + y^2 + z^2 = 1,$$

with parameterization

$$X(x, y) = (\cos x, \sin x \cos y, \sin x \sin y),$$

which implies that

$$X_x = (-\sin x, \cos x \cos y, \cos x \sin y),$$

$$X_y = (0, -\sin x \sin y, \sin x \cos y).$$

The components of the first fundamental form are given by

$$E = X_x \cdot X_x = (-\sin x, \cos x \cos y, \cos x \sin y) \cdot (-\sin x, \cos x \cos y, \cos x \sin y) = 1,$$

$$F = X_x \cdot X_y = (-\sin x, \cos x \cos y, \cos x \sin y) \cdot (0, -\sin x \sin y, \sin x \cos y) = 0,$$

$$G = X_y \cdot X_y = (0, -\sin x \sin y, \sin x \cos y) \cdot (0, -\sin x \sin y, \sin x \cos y) = \sin^2 x.$$

So, the metric is obviously

$$g = dx^2 + \sin^2 x dy^2 = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2 x \end{bmatrix},$$

and

$$det(g) = \sin^2 x,$$

which implies that

$$g^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \csc^2 x \end{bmatrix}.$$

So, using equation (1.1), the Laplacian is given by

$$\begin{aligned}
\triangle u &= \tfrac{1}{\sqrt{|\det(g)|}} \tfrac{\partial}{\partial x^i} \left(\sqrt{|\det(g)|} g^{ij} \tfrac{\partial u}{\partial x^j} \right), \\
&= \tfrac{1}{\sin x} \tfrac{\partial}{\partial x} \left(1 \cdot \sin x \tfrac{\partial u}{\partial x} + 0 \cdot \sin x \tfrac{\partial u}{\partial y} \right) + \tfrac{1}{\sin x} \tfrac{\partial}{\partial y} \left(0 \cdot \sin x \tfrac{\partial u}{\partial x} + \csc^2 x \cdot \sin x \tfrac{\partial u}{\partial y} \right), \\
&= \tfrac{1}{\sin x} \tfrac{\partial}{\partial x} \left(\sin x \tfrac{\partial u}{\partial x} \right) + \tfrac{1}{\sin x} \tfrac{\partial}{\partial y} \left(\csc x \tfrac{\partial u}{\partial y} \right), \\
&= u_{xx} + (\cot x) u_x + \left(\csc^2 x \right) u_{yy}.
\end{aligned} \tag{1.3}$$

Hence, the **nonlinear** Klein Gordon equation becomes of the form

$$u_{tt} = u_{xx} + (\cot x) u_x + \left(\csc^2 x \right) u_{yy} + f(u), \tag{1.4}$$

where $f(u)$ is a nonlinear function.

For the linear case, we will consider the **linear** Klein Gordon equation of the form

$$u_{tt} = u_{xx} + (\cot x)\, u_x + \left(\csc^2 x\right) u_{yy} + au, \tag{1.5}$$

where $a \neq 0$.

CHAPTER 2
Lie Symmetry method for differential equations

This chapter focuses on basic ideas of Lie symmetry method that should serve the bases for the research results presented in chapters 3 and 4. The main aim is to give a short review of the standard background in Lie symmetry method for ODEs and PDEs. Since the fundamental results of Lie symmetry methods are well established and have become a standard now, most of the proofs in this short review are omitted. However, the necessary details are presented where these were thought to be essential. This review follows the books [8, 21]. In general, the reader is referred to the standard books [8, 17, 21, 31, 36] for thorough introduction and complete understanding of the subject of Lie symmetry analysis and its applications.

Section 2.1 and section 2.2 present the concepts and methods of Lie symmetries needed to deal with ODEs and PDEs respectively.

2.1 Lie symmetry method for ODEs

Beginning with the ideas of one parameter groups and their infinitesimal generators, this section presents the notions of prolongations, symmetry of ODEs as well as the method of finding symmetries of ODEs. The section ends with illustrative examples of reduction of order of ODEs using symmetries.

2.1.1 One parameter group of transformations and infinitesimal generators

The study of Lie symmetries of ODEs involves one parameter groups of transformations in plane.

Definition: (Point transformation in xy−plane)

A point transformation in $xy-$plane is a function

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,$$

given by

$$(x, y) \xrightarrow{T} (\overline{x}, \overline{y}),$$

where

$$\overline{x} = f(x, y),$$

$$\overline{y} = g(x, y).$$

A transformation in the $xy-$plane which transforms a point (x, y) to another point $(\overline{x}, \overline{y})$.

Examples of transformations in the $xy-$plane:

Translation:

$$\overline{x} = x + 3,$$

$$\overline{y} = y - 2.$$

Dilation:

$$\overline{x} = 2x,$$

$$\overline{y} = 5y.$$

Rotation:

$$\overline{x} = x\cos\theta - y\sin\theta,$$

$$\overline{y} = x\sin\theta + y\cos\theta.$$

Definition: (one parameter transformation)

A one parameter transformation is a transformation that depends on one parameter only. If the parameter is ϵ, then it is of the form

$$\overline{x} = f\left(x, y, \epsilon\right),$$

$$\overline{y} = g\left(x, y, \epsilon\right).$$

Examples:

1) The rotation above depends only on the parameter θ.

2) The translation:

$$\overline{x} = x + \epsilon,$$

$$\overline{y} = y + 2\epsilon.$$

is a one parameter transformation depending on ϵ only.

3) The translation

$$\overline{x} = x + k_1,$$

$$\overline{y} = y + k_2.$$

is **not** one parameter transformation since it is associated with **two parameters** k_1 and k_2.

Definition: (one parameter group of transformations)

The one parameter transformation

$$\overline{x} = f\left(x, y, \epsilon\right),$$

$$\overline{y} = g\left(x, y, \epsilon\right).$$

is a one parameter group of transformations if the following properties hold:

(i) Identity:

The transformation with $\epsilon = 0$ is the identity transformation.

$$T_0(x, y) = (x, y).$$

(ii) Closure:

The composition of two transformations in the group is a member of the set of transformation in the group

$$T_a T_b = T_c.$$

(iii) Inverse:

The transformation with $-\epsilon$ gives the inverse transformation

$$T_\epsilon T_{-\epsilon} = T_{-\epsilon} T_\epsilon = T_0.$$

Remark: the associativity holds always because the composition operation is always associative.

Example: Show that the following transformation is a one parameter group.

$T_\epsilon : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$\overline{x} = x + \epsilon,$$

$$\overline{y} = y + 2\epsilon.$$

Solution:

(i) The identity transformation with $\epsilon = 0$,

$$T_0(x, y) = (x, y).$$

(ii) Let $a, b \in \mathbb{R}$,

$$T_b T_a(x, y) = T_b(x + a, y + 2a) = (x + a + b, y + 2a + 2b) = (x + (a + b), y + 2(a + b)),$$
$$= T_{a+b}(x, y).$$

So, the set of these transformations is closed.

(iii) The inverse of T_ϵ is

$$T_\epsilon^{-1} = T_{-\epsilon},$$

because

$$T_\epsilon T_{-\epsilon} = T_{\epsilon-\epsilon} = T_0,$$

and

$$T_{-\epsilon} T_\epsilon = T_{-\epsilon+\epsilon} = T_0.$$

Infinitesimal generators of one parameter group of transformations

Given a one parameter group of transformations

$$\overline{x} = f(x, y, \epsilon), \tag{2.1}$$

$$\overline{y} = g(x, y, \epsilon), \tag{2.2}$$

with

$$f(x,y,0) = x, \tag{2.3}$$

$$g(x,y,0) = y. \tag{2.4}$$

This obviously defines a curve $\alpha(\epsilon) = (\overline{x}, \overline{y}) = (f(x,y,\epsilon), g(x,y,\epsilon))$.

If the tangent vector to the curve through the point (x,y) is denoted by (ξ, η), then the operator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}, \tag{2.5}$$

is called **infinitesimal generator** of one parameter group of transformations. This gives directions in which the point will move along the curve

$$\alpha(\epsilon) = (\overline{x}, \overline{y}) = (f(x,y,\epsilon), g(x,y,\epsilon)).$$

For practical purpose, it is more convenient to use the infinitesimal generator (or the operator form) of the group. So it is important to learn how to find the generator from a group, and how to find the group from a generator.

Infinitesimal generator of groups

Given a one parameter group of transformations

$$\overline{x} = f(x,y,\epsilon), \tag{2.6}$$

$$\overline{y} = g(x,y,\epsilon), \tag{2.7}$$

with

$$f(x,y,0) = x, \tag{2.8}$$

$$g(x,y,0) = y. \tag{2.9}$$

The associated infinitesimal generator is given by

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}, \tag{2.10}$$

where

$$\xi(x,y) = \frac{df(x,y,\epsilon)}{d\epsilon}, \tag{2.11}$$

$$\eta(x,y) = \frac{dg(x,y,\epsilon)}{d\epsilon}. \tag{2.12}$$

Example: Find the infinitesimal generator of the group of rotations

$$\overline{x} = x\cos\theta - y\sin\theta, \tag{2.13}$$

$$\overline{y} = x\sin\theta + y\cos\theta. \tag{2.14}$$

Solution:

$$\xi(x,y) = \frac{d\overline{x}}{d\theta} = -x\sin 0 - y\cos 0 = -y,$$

$$\eta(x,y) = \frac{d\overline{y}}{d\theta} = x\cos 0 - y\sin 0 = x.$$

The infinitesimal generator associated with this group then is given by

$X=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}.$

Some important groups with their associated infinitesimals are given in the table below

Table 1

Commutator table for the Lie algebra

Action	Group	infinitesimal Generator
Translation in $x-axis$	$\overline{x}=x+\epsilon,\ \ \overline{y}=y$	$\frac{\partial}{\partial x}$
Translation in $y-axis$	$\overline{x}=x,\ \ \overline{y}=y+\epsilon$	$\frac{\partial}{\partial y}$
Dilation in $x-axis$	$\overline{x}=e^{\epsilon}x,\ \ \overline{y}=y$	$x\frac{\partial}{\partial x}$
Dilation in $y-axis$	$\overline{x}=x,\ \ \overline{y}=e^{\epsilon}y$	$y\frac{\partial}{\partial y}$
Irregular Dilation	$\overline{x}=e^{a\epsilon}x,\ \ \overline{y}=e^{b\epsilon}y$	$ax\frac{\partial}{\partial x}+by\frac{\partial}{\partial y}$
Rotation	$\overline{x}=x\cos\theta-y\sin\theta,\ \ \overline{y}=x\sin\theta+y\cos\theta$	$-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}$
	$\overline{x}=\frac{x}{1-\epsilon y},\ \ \overline{y}=\frac{y}{1-\epsilon y}$	$xy\frac{\partial}{\partial x}+y^2\frac{\partial}{\partial y}$

Group corresponding to generator

Given infinitesimal generator

$$X=\xi\left(x,y\right)\frac{\partial}{\partial x}+\eta\left(x,y\right)\frac{\partial}{\partial y}. \tag{2.15}$$

The associated one parameter group of transformations

$$\overline{x}=f\left(x,y,\epsilon\right), \tag{2.16}$$

$$\overline{y}=g\left(x,y,\epsilon\right), \tag{2.17}$$

with

$$f\left(x,y,0\right)=x, \tag{2.18}$$

$$g\left(x,y,0\right)=y, \tag{2.19}$$

can be found by solving the system of ODEs given by

$$\frac{d\overline{x}}{d\epsilon}=\xi\left(\overline{x},\overline{y}\right), \tag{2.20}$$

$$\frac{d\overline{y}}{d\epsilon}=\eta\left(\overline{x},\overline{y}\right), \tag{2.21}$$

with initial conditions

$$\overline{x}=x, \tag{2.22}$$

$$\overline{y}=y. \tag{2.23}$$

Example: Find the one parameter group corresponding to the generator

$$X=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}. \tag{2.24}$$

Solution: Need to solve the system

$$\frac{d\overline{x}}{d\epsilon}=\overline{x}, \tag{2.25}$$

$$\frac{d\overline{y}}{d\epsilon}=\overline{y}, \tag{2.26}$$

with the initial conditions

$$\overline{x}\left(0\right)=x, \tag{2.27}$$

$$\overline{y}\left(0\right)=y. \tag{2.28}$$

Solving equation (2.25) gives

$$\overline{x}=Ce^{\epsilon}. \tag{2.29}$$

Applying the initial condition (2.27) implies

$$C=x. \tag{2.30}$$

Therefore,

$$\overline{x}=xe^{\epsilon}. \tag{2.31}$$

Similarly, solving equation (2.26) with the initial condition (2.28) gives

$$\overline{y}=ye^{\epsilon}. \tag{2.32}$$

2.1.2 Definition of symmetries of ODEs

Given an ODE

$$F\left(x,y,y^{'},....,y^{(n)}\right)=0, \tag{2.33}$$

A one parameter group of transformations

$$\overline{x}=f\left(x,y,\epsilon\right), \tag{2.34}$$

$$\overline{y}=g\left(x,y,\epsilon\right), \tag{2.35}$$

is called a **symmetry** of ODE (2.33) if the form of the ODE (2.33) remains unchanged under the transformation (2.34) and (2.35).

i.e. after change of variables (x,y) to $(\overline{x},\overline{y})$, we get

$$F\left(\overline{x},\overline{y},\overline{y^{'}},....,\overline{y^{(n)}}\right)=0. \tag{2.36}$$

Example: Show that the transformation

$$\overline{x}=xe^{\epsilon}, \tag{2.37}$$

$$\overline{y}=ye^{\epsilon}, \tag{2.38}$$

is a symmetry of the ODE

$$\frac{dy}{dx}=f\left(\frac{y}{x}\right). \tag{2.39}$$

Solution:

Equation (2.37) implies

$$d\overline{x}=e^{\epsilon}dx. \tag{2.40}$$

and equation (2.38) implies

$$d\overline{y} = e^{\epsilon} dy. \tag{2.41}$$

Dividing equation (2.41) over equation (2.40) gives

$$\frac{d\overline{y}}{d\overline{x}} = \frac{e^{\epsilon} dy}{e^{\epsilon} dx} = \frac{dy}{dx}. \tag{2.42}$$

Using ODE (2.39), we get

$$\frac{d\overline{y}}{d\overline{x}} = f\left(\frac{y}{x}\right) = f\left(\frac{e^{-\epsilon} y}{e^{-\epsilon} x}\right) = f\left(\frac{\overline{y}}{\overline{x}}\right). \tag{2.43}$$

which is of the same form as the original ODE.

This implies that the transformation forms a symmetry for the ODE.

2.1.3 Prolongation of infinitesimal generators

Let

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}. \tag{2.44}$$

Then, using Taylor's series, the infinitesimal transformation of the corresponding group is given by

$$\overline{x} = x + \epsilon \xi(x, y) + O\left(\epsilon^2\right), \tag{2.45}$$

$$\overline{y} = y + \epsilon \eta(x, y) + O\left(\epsilon^2\right). \tag{2.46}$$

In order to study symmetries of ODEs, we need to know how the derivatives are transformed via the extension of above transformation.

First prolongation

We look at the transformation of the derivative

$$y^{'} = \frac{dy}{dx} \longrightarrow \overline{y^{'}} = \frac{d\overline{y}}{d\overline{x}}.$$

Let

$$\overline{y^{'}} = y^{'} + \epsilon \eta^{[1]}\left(x, y, y^{'}\right) + O\left(\epsilon^2\right). \tag{2.47}$$

Let us find $\eta^{[1]}\left(x, y, y^{'}\right)$,

$$\begin{aligned}
\overline{y^{'}} &= \frac{d\overline{y}}{d\overline{x}} = \frac{dy + \epsilon d\eta + O\left(\epsilon^2\right)}{dx + \epsilon d\xi + O(\epsilon^2)} = \frac{\frac{dy}{dx} + \epsilon \frac{d\eta}{dx}}{1 + \epsilon \frac{d\xi}{dx}}, \\
&= \left(y^{'} + \epsilon \frac{d\eta}{dx}\right)\left(1 + \epsilon \frac{d\xi}{dx}\right)^{-1}, \\
&= \left(y^{'} + \epsilon \frac{d\eta}{dx}\right)\left(1 - \epsilon \frac{d\xi}{dx} + O\left(\epsilon^2\right)\right), \\
&= y^{'} + \epsilon\left(\frac{d\eta}{dx} - y^{'} \frac{d\xi}{dx}\right) + O\left(\epsilon^2\right).
\end{aligned}$$

This implies that

$$\eta^{[1]} = \frac{d\eta}{dx} - y^{'} \frac{d\xi}{dx}, \tag{2.48}$$

or

$$\eta^{[1]} = \eta_x - y^{'} \eta_y - y^{'}\left(\xi_x + y^{'} \xi_y\right),$$

or

$$\eta^{[1]} = \eta_x + \left(\eta_y - \xi_x\right) y^{'} - \xi_y y^{'^2}. \tag{2.49}$$

Thus, the first prolongation is given by

$$X^{[1]} = \xi\left(x,y\right)\frac{\partial}{\partial x} + \eta\left(x,y\right)\frac{\partial}{\partial y} + \left[\eta_x + \left(\eta_y - \xi_x\right) y^{'} - \xi_y y^{'^2}\right]\frac{\partial}{\partial y^{\prime}}, \tag{2.50}$$

which gives infinitesimal transformation for $x, y, y^{'}$.

Second prolongation

Let

$$X = \xi\left(x,y\right)\frac{\partial}{\partial x} + \eta\left(x,y\right)\frac{\partial}{\partial y},$$

be infinitesimal transformation and

$$X^{[1]} = \xi\left(x,y\right)\frac{\partial}{\partial x} + \eta\left(x,y\right)\frac{\partial}{\partial y} + \eta^{[1]}\left(x,y,y^{'}\right)\frac{\partial}{\partial y^{\prime}},$$

be the its first prolongation. Then, the transformation of x, y and $y^{'}$are given by

$$\overline{x} = x + \epsilon\xi\left(x,y\right) + O\left(\epsilon^2\right),$$
$$\overline{y} = y + \epsilon\eta\left(x,y\right) + O\left(\epsilon^2\right),$$
$$\overline{y^{'}} = y^{'} + \epsilon\eta^{[1]}\left(x,y,y^{'}\right) + O\left(\epsilon^2\right).$$

We want to find the transformation of $y^{''}$,

$$y^{''} = \frac{d^2y}{dx^2} \longrightarrow \overline{y^{''}} = \frac{d\overline{y^{'}}}{d\overline{x}}.$$

Let

$$\overline{y^{''}} = y^{''} + \epsilon\eta^{[2]}\left(x,y,y^{'},y^{''}\right) + O\left(\epsilon^2\right).$$

Let us find $\eta^{[2]}\left(x,y,y^{'},y^{''}\right)$. The transformation of $y^{''}$ is given by

$$\begin{aligned}
\overline{y^{''}} &= \frac{d\overline{y^{'}}}{d\overline{x}} = \frac{dy^{'} + \epsilon d\eta^{[1]} + O\left(\epsilon^2\right)}{dx + \epsilon d\xi + O\left(\epsilon^2\right)} = \frac{\frac{dy^{'}}{dx} + \epsilon\frac{d\eta^{[1]}}{dx}}{1 + \epsilon\frac{d\xi}{dx}},\\
&= \left(y^{''} + \epsilon\frac{d\eta^{[1]}}{dx}\right)\left(1 + \epsilon\frac{d\xi}{dx}\right)^{-1},\\
&= \left(y^{''} + \epsilon\frac{d\eta^{[1]}}{dx}\right)\left(1 - \epsilon\frac{d\xi}{dx} + O\left(\epsilon^2\right)\right),\\
&= y^{''} + \epsilon\left(\frac{d\eta^{[1]}}{dx} - y^{''}\frac{d\xi}{dx}\right) + O\left(\epsilon^2\right).
\end{aligned}$$

This implies

$$\eta^{[2]} = \frac{d\eta^{[1]}}{dx} - y^{''}\frac{d\xi}{dx}, \tag{2.51}$$

or

$$\begin{aligned}
\eta^{[2]} &= \frac{d\left[\eta_x + \left(\eta_y - \xi_x\right)y^{'} - \xi_y y^{'^2}\right]}{dx} - y^{''}\frac{d\xi}{dx},\\
&= \eta_{xx} + y^{'}\eta_{xy} + \left(\eta_{yx} + y^{'}\eta_y - \xi_{xx} - y^{'}\xi_{xy}\right)y^{'} + \left(\eta_y - \xi_x\right)y^{''}\\
&\quad - \left(\xi_{yx} + y^{'}\xi_{yy}\right)y^{'^2} - 2y^{'}y^{''}\xi_y - \left(\xi_x + y^{'}\xi_y\right)y^{''}.
\end{aligned}$$

Simplifying gives

$$\eta^{[2]} = \eta_{xx} + \left(2\eta_{xy} - \xi_{xx}\right)y^{'} + \left(\eta_{yy} - 2\xi_{xy}\right)y^{'^2} - \xi_{yy}y^{'^3} + \left(\eta_y - 2\xi_x\right)y^{''} - 3y^{'}y^{''}\xi_y. \tag{2.52}$$

Thus, the second prolongation is given by

$$X^{[2]} = \xi\left(x,y\right)\frac{\partial}{\partial x} + \eta\left(x,y\right)\frac{\partial}{\partial y} + \eta^{[1]}\left(x,y,y^{'}\right)\frac{\partial}{\partial y^{'}} + \eta^{[2]}\left(x,y,y^{'},y^{''}\right)\frac{\partial}{\partial y^{''}}. \tag{2.53}$$

The nth prolongation

In general, we can show by induction that

$$\eta^{[n]} = \frac{d\eta^{[n-1]}}{dx} - y^{(n)}\frac{d\xi}{dx} \text{ for } n = 1,2,3,....., \tag{2.54}$$

where

$$\eta^{[0]} = \eta,$$

and from this

$$X^{[n]} = \xi\left(x,y\right)\frac{\partial}{\partial x} + \eta\left(x,y\right)\frac{\partial}{\partial y} + \sum_{k=1}^{n}\eta^{[k]}\frac{\partial}{\partial y^{(k)}}. \tag{2.55}$$

Example: Find the second prolongation of $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.

Solution:

$$\xi\left(x,y\right) = x, \tag{2.56}$$

$$\eta\left(x,y\right) = y. \tag{2.57}$$

Then,

$$\eta^{[1]} = \eta_x + \left(\eta_y - \xi_x\right)y^{'} - \xi_y y^{'2} = 0, \tag{2.58}$$

$$\begin{aligned}\eta^{[2]} &= \eta_{xx} + \left(2\eta_{xy} - \xi_{xx}\right)y^{'} + \left(\eta_{yy} - 2\xi_{xy}\right)y^{'2} - \xi_{yy}y^{'3} + \left(\eta_y - 2\xi_x\right)y^{''} - 3y^{'}y^{''}\xi_y,\\ &= -y^{''}.\end{aligned} \tag{2.59}$$

Thus,

$$\begin{aligned}X^{[2]} &= \xi\left(x,y\right)\frac{\partial}{\partial x} + \eta\left(x,y\right)\frac{\partial}{\partial y} + \eta^{[1]}\left(x,y,y^{'}\right)\frac{\partial}{\partial y^{'}} + \eta^{[2]}\left(x,y,y^{'},y^{''}\right)\frac{\partial}{\partial y^{''}},\\ &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 0\frac{\partial}{\partial y^{'}} - y^{''}\frac{\partial}{\partial y^{''}}.\end{aligned} \tag{2.60}$$

2.1.4 Invariance criteria for finding the symmetries of ODEs

Given an n^{th} order ODE

$$F\left(x,y,y^{'},....,y^{(n)}\right) = 0, \tag{2.61}$$

A symmetry of ODE (2.61) is a one parameter group infinitesimal transformations

$$\overline{x} = x + \epsilon\xi\left(x,y\right), \tag{2.62}$$

$$\overline{y} = y + \epsilon\eta\left(x,y\right), \tag{2.63}$$

that leaves the ODE (2.61) invariant, i. e.

$$F\left(\overline{x},\overline{y},\overline{y^{'}},....,\overline{y^{(n)}}\right) = 0, \tag{2.64}$$

where $\overline{y^{'}},....,\overline{y^{(n)}}$ are transformed through prolongations of the group.

Treating equation (2.61) as an algebraic equation in $x,y,y^{'},....,y^{(n)}$, invariance criteria can be generalized to the following definition.

Definition:

The infinitesimal transformation

$$\overline{x} = x + \epsilon\xi\left(x,y\right), \tag{2.65}$$

$$\overline{y} = y + \epsilon\eta\left(x,y\right), \tag{2.66}$$

or equivalently the generator

$$X = \xi\left(x,y\right)\frac{\partial}{\partial x} + \eta\left(x,y\right)\frac{\partial}{\partial y}, \tag{2.67}$$

is a **symmetry** of ODE (2.61) if

$$X^{[n]}F = 0, \tag{2.68}$$

whenever

$$F = 0,$$

where $X^{[n]}$ is the n^{th} prolongation of X.

Example: Show that $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ is a symmetry of the ODE

$$x^2y^{''} + xy^{'2} - yy^{'} = 0. \tag{2.69}$$

Solution:

The function associated with the ODE is

$$F\left(x,y,y^{'},y^{''}\right) = x^2y^{''} + xy^{'2} - yy^{'}. \tag{2.70}$$

The second prolongation of X as found in the previous example is given by

$$X^{[2]} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 0\frac{\partial}{\partial y^{'}} - y^{''}\frac{\partial}{\partial y^{''}}. \tag{2.71}$$

Hence,

$$\begin{aligned}
X^{[2]}F|_{F=0} &= x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + 0\frac{\partial F}{\partial y^{'}} - y^{''}\frac{\partial F}{\partial y^{''}}, \\
&= x\left(2xy^{''} + y^{'2}\right) + y\left(-y^{'}\right) - y^{''}\left(x^2\right), \\
&= 2x^2y^{''} + xy^{'2} - y^{'}y - x^2y^{''}, \\
&= x^2y^{''} + xy^{'2} - yy^{'}, \\
&= 0.
\end{aligned}$$

This implies that

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

is a **symmetry** of the given ODE.

2.1.5 Method of finding symmetries of ODEs

The criterion

$$X^{[n]}F|_{F=0} = 0, \tag{2.72}$$

is called the invariance criteria for the symmetries of ODE (2.61) of order n. It is the basis for computing the symmetries of ODE (2.61). This will lead us to an over determined system of linear PDEs in $\xi(x,y)$ and $\eta(x,y)$. Solving this system will give us all possible functions $\xi(x,y)$ and $\eta(x,y)$ that satisfy the invariance condition.

Steps for finding symmetries of ODE (2.61)

1) Find prolongation $X^{[n]}$ where n is the order of the ODE.

2) Apply the prolongation to the ODE restricted to the validation of ODE.

3) Obtain determining equations from step 2 by comparing coefficients of powers of derivatives.

4) Simplify and solve the determining equations.

Example: (Simple example to show the procedure, the same procedure works for any other ODE)

Find all symmetries of

$$y^{''} = 0.$$

Solution: Let

$$F\left(x, y, y^{'}, y^{''}\right) = y^{''}. \tag{2.73}$$

Let the symmetry be of the form

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}. \tag{2.74}$$

Step 1: write the 2^{nd} prolongation

$$X^{[2]} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \eta^{[1]}\frac{\partial}{\partial y^{'}} + \eta^{[2]}\frac{\partial}{\partial y^{''}}. \tag{2.75}$$

Step 2 : Apply the 2^{nd} prolongation to the differential equation (2.73)

$$X^{[2]}\left(y^{''}\right) = \eta_{xx} + \left(2\eta_{xy} - \xi_{xx}\right)y^{'} + \left(\eta_{yy} - 2\xi_{xy}\right)y^{'^{2}} - \xi_{yy}y^{'^{3}}. \tag{2.76}$$

Step 3 : Finding the determining equations as follows

$$\text{Coefficient of } \left(y^{'}\right)^{3} : \xi_{yy} = 0, \tag{2.77}$$

$$\text{Coefficient of } \left(y^{'}\right)^{2} : \eta_{yy} - 2\xi_{xy} = 0, \tag{2.78}$$

$$\text{Coefficient of } \left(y^{'}\right)^{1} : 2\eta_{xy} - \xi_{xx} = 0, \tag{2.79}$$

$$\text{Coefficient of } \left(y^{'}\right)^{0} : \eta_{xx} = 0. \tag{2.80}$$

We get a system of linear PDEs.

Step 4: Solving the over determined system of linear PDEs (2.77) - (2.80)

From equation (2.77), ξ is linear in y. So,

$$\xi = A(x)\,y + B(x)\,. \tag{2.81}$$

From equation (2.80), η is linear in y. So,

$$\eta = f(y)\,x + g(y)\,. \tag{2.82}$$

Differentiating equation (2.78) with respect to y,

$$\eta_{yyy} - 2\xi_{xyy} = 0. \tag{2.83}$$

Using equation (2.77) in equation (2.83), gives

$$\eta_{yyy} = 0, \tag{2.84}$$

which implies

$$f^{'''}(y)\,x + g^{'''}(y) = 0,$$

or

$$f^{'''}(y) = 0, \tag{2.85}$$

and

$$g^{'''}(y) = 0. \tag{2.86}$$

From equation (2.85), we get

$$f(y) = c_1 + c_2 y + c_{10} y^2. \tag{2.87}$$

From equation (2.86), we get

$$g(y) = c_4 + c_5 y + c_6 y^2. \tag{2.88}$$

Substituting the functions $f(y)$ and $g(y)$ in equation (2.82) gives

$$\eta = \left(c_1 + c_2 y + c_{10} y^2\right) x + \left(c_4 + c_5 y + c_6 y^2\right). \tag{2.89}$$

Substituting equations (2.89) and (2.81) in equation (2.78), we get

$$2\left(c_2 + 2c_{10} y\right) - \left(A^{''}(x)\,y + B^{''}(x)\right) = 0. \tag{2.90}$$

Comparing coefficients of powers of y in equation (2.90) gives

$$A^{''}(x) = 4c_{10}, \tag{2.91}$$

$$B^{''}(x) = 2c_2. \tag{2.92}$$

Integrating equation (2.91), we get

$$A(x) = c_7 + c_9 x + 2c_3 x^2. \tag{2.93}$$

And integrating equation (2.92), we get

$$B(x) = c_8 + c_3 x + c_2 x^2. \tag{2.94}$$

Substituting equations (2.93) and (2.94) in equation (2.81) gives

$$\xi = \left(c_7 + c_9 x + 2c_3 x^2\right) y + \left(c_8 + c_3 x + c_2 x^2\right). \tag{2.95}$$

Substituting equations (2.89) and (2.95) in equation (2.78), we get

$$2c_{10} x + 2c_6 = 2c_9 + 8c_{10} x, \tag{2.96}$$

which implies

$$c_9 = c_6, \tag{2.97}$$

$$c_{10} = 0. \tag{2.98}$$

Remove c_9 and c_{10} from the formulas

$$\xi = (c_7 + c_6 x)\, y + \left(c_8 + c_3 x + c_2 x^2\right), \tag{2.99}$$

$$\eta = (c_1 + c_2 y)\, x + \left(c_4 + c_5 y + c_6 y^2\right). \tag{2.100}$$

So, we obtain 8 independent constants (parameters). This provides 8 dimensional Lie algebra generated by the symmetries found below.

Putting $c_1 = 1$ and the remaining constants vanish implies

$$X_1 = x\frac{\partial}{\partial y}. \tag{2.101}$$

Putting $c_2 = 1$ and the remaining constants vanish implies

$$X_2 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}. \tag{2.102}$$

Putting $c_3 = 1$ and the remaining constants vanish implies

$$X_3 = x\frac{\partial}{\partial x}. \tag{2.103}$$

Putting $c_4 = 1$ and the remaining constants vanish implies

$$X_4 = \frac{\partial}{\partial y}. \tag{2.104}$$

Putting $c_5 = 1$ and the remaining constants vanish implies

$$X_5 = y\frac{\partial}{\partial y}. \tag{2.105}$$

Putting $c_6 = 1$ and the remaining constants vanish implies

$$X_6 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}. \tag{2.106}$$

Putting $c_7 = 1$ and the remaining constants vanish implies

$$X_7 = y\frac{\partial}{\partial x}. \tag{2.107}$$

Putting $c_8 = 1$ and the remaining constants vanish implies

$$X_8 = \frac{\partial}{\partial x}. \tag{2.108}$$

Note: the set of all symmetries of an ODE forms a Lie Algebra with commutator operation

$$[X, Y] = XY - YX,$$

We obtain the basis of all symmetries

$$S = \{X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, \},$$

with commutation relations

Table 2

Commutator table for the Lie algebra

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	0	$-X_1$	0	X_1	X_2	$X_3 - X_6$	$-X_4$
X_2	0	0	$-X_2$	$-X_1$	0	0	$-X_6$	$-2X_3 - X_5$
X_3	X_1	X_2	0	0	0	0	$-X_7$	$-X_8$
X_4	0	X_1	0	0	X_4	$X_3 + 2X_5$	X_8	0
X_5	$-X_1$	0	0	$-X_4$	0	X_6	X_7	0
X_6	$-X_2$	0	0	$-X_3 - 2X_5$	$-X_6$	0	0	$-X_7$
X_7	$X_6 - X_3$	X_6	X_7	$-X_8$	$-X_7$	0	0	0
X_8	X_4	$2X_3 + X_5$	X_8	0	0	X_7	0	0

.

2.1.6 Reduction of order of ODEs using symmetries

Given a symmetry

$$X = \xi\left(x,y\right)\frac{\partial}{\partial x} + \eta\left(x,y\right)\frac{\partial}{\partial y}, \tag{2.109}$$

of an ODE

$$F\left(x,y,y^{'},....,y^{(n)}\right) = 0. \tag{2.110}$$

The invariants obtained through

$$X^{[n]}I = 0, \tag{2.111}$$

can be utilized, cf. [8], to reduce the order of ODE by the standard procedure of change of variables. This method is illustrated in the examples below.

Example: Reduce

$$F = x^2y^{''} + xy^{'2} - yy^{'} = 0, \tag{2.112}$$

using the symmetry

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{2.113}$$

Solution:

The 2^{nd} prolongation of X is

$$X^{[2]} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 0\frac{\partial}{\partial y'} - y^{''}\frac{\partial}{\partial y''}. \tag{2.114}$$

We find invariants using $X^{[n]}I = 0$, whose characteristic system is

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dy^{'}}{0} = \frac{dy^{''}}{-y^{''}}.$$

Solving $\frac{dx}{x} = \frac{dy}{y}$ implies

$$\frac{y}{x} = c.$$

Solving $\frac{dx}{x} = \frac{dy^{'}}{0}$ implies

$$y^{'} = c.$$

Thus, we obtain two invariants of $X^{[2]}$ given by

$$u = \frac{y}{x}, \tag{2.115}$$

$$v = y^{'}. \tag{2.116}$$

To write the given ODE in terms of u and v,

$$x^2 y^{''} = x^2 \frac{dy^{'}}{dx} = x^2 \frac{dv}{dx} = x^2 \frac{dv}{du}\frac{du}{dx} = x^2 \frac{dv}{du}\frac{x\frac{dy}{dx} - y}{x^2} = \frac{dv}{du}\left(xv - y\right). \tag{2.117}$$

Using equations (2.115) - (2.117) in the ODE (2.112) gives

$$x^2 y^{''} + xy^{'2} - yy^{'} = (xv - y)\frac{dv}{du} + xv^2 - yv = 0. \tag{2.118}$$

Dividing equation (2.118) by x implies

$$(v - u)\frac{dv}{du} + v^2 - uv = 0, \tag{2.119}$$

(i) If $u \neq v$, we get

$$\frac{dv}{du} = -v, \tag{2.120}$$

which is a separable 1^{st} order ODE with solution

$$v = ce^{-u}. \tag{2.121}$$

Going back to the original variables gives

$$\frac{dy}{dx} = ce^{-\frac{y}{x}}, \tag{2.122}$$

which is a homogeneous first order ODE.

Let $V = \frac{y}{x}$, then

$$\frac{dy}{dx} = \frac{xdV + Vdx}{dx} = x\frac{dV}{dx} + V. \tag{2.123}$$

Substituting this in ODE (2.122) leads to a separable 1^{st} order ODE

$$x\frac{dV}{dx} + V = ce^{-V},$$

or

$$\frac{dV}{ce^{-V} - V} = \frac{dx}{x}. \tag{2.124}$$

(ii) If $u = v$, then

$$y^{'} = \frac{y}{x}, \tag{2.125}$$

which implies

$$\frac{dy}{y} = \frac{dx}{x}, \tag{2.126}$$

which gives the solution

$$y = ax.$$

Example: Reduce the following ODE using symmetry technique

$$xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 + y\frac{dy}{dx} - xy^2 = 0. \tag{2.127}$$

Solution:

We can check that

$$X = y\frac{\partial}{\partial y}, \tag{2.128}$$

is a symmetry of ODE (2.127).

Next we use the symmetry to reduce the ODE (2.127). The 2^{nd} prolongation of X is

$$X^{[2]} = 0\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + y^{'}\frac{\partial}{\partial y'} + y^{''}\frac{\partial}{\partial y''}. \tag{2.129}$$

Solving the characteristic system of $X^{[n]}I = 0$ gives differential invariants:

$$u = x, \tag{2.130}$$

$$v = \frac{y^{'}}{y}. \tag{2.131}$$

These variables give

$$y^{''} = \frac{dy^{'}}{dx} = \frac{d(yv)}{dx} = y\frac{dv}{du} + v\frac{dy}{dx} = y\frac{dv}{du} + v^2 y, \tag{2.132}$$

or

$$\frac{y^{''}}{y} = \frac{dv}{du} + v^2. \tag{2.133}$$

Putting this in equation (2.127) reduces it to the first order ODE

$$xy^2\left(\frac{dv}{du} + v^2\right) + xy^2v^2 + y^2v - xy^2 = 0. \tag{2.134}$$

Dividing equation (2.134) by xy^2 leads to

$$u\left(\frac{dv}{du} + v^2\right) + uv^2 + v - u = 0, \tag{2.135}$$

or

$$\frac{dv}{du} = -2v^2 - \frac{1}{u}v + 1. \tag{2.136}$$

The reduced 1^{st} order ODE is in the form of Riccati equation.

2.2 Lie symmetry method for PDEs in three independent variables

This section is devoted to the discussion of symmetries of PDEs, the prolongations of their generators and the method of finding symmetries of PDEs. For the sake of clarity and without loss of generality, we restrict the main discussions to 2^{nd} order PDEs with one dependent variable u and three independent variables x,y,t. This does not hamper the research work in the thesis since the PDEs involved in the research work in chapters 3 and 4 all belong to this class of PDEs. A detailed account of the subject of Lie symmetry for general PDEs is contained in many standard books on the topic cf. [2, 6, 7, 8, 11, 15, 17, 18, 19, 20, 30, 31, 32, 36].

2.2.1 Symmetries of PDEs

Consider a PDE

$$F\left(x,y,t,u,u_x,u_y,u_t,u_{xx},u_{yy},u_{tt},u_{xy},u_{xt},u_{yt}\right)=0, \tag{2.137}$$

A one parameter group of transformations

$$\overline{x}=f\left(x,y,t,u,\epsilon\right), \tag{2.138}$$

$$\overline{y}=g\left(x,y,t,u,\epsilon\right), \tag{2.139}$$

$$\overline{t}=h\left(x,y,t,u,\epsilon\right), \tag{2.140}$$

$$\overline{u}=j\left(x,y,t,u,\epsilon\right), \tag{2.141}$$

is called a **symmetry** of PDE (2.137) if the form of PDE (2.137) remains unchanged under the transformations (2.138) - (2.141), i.e. after change of variables

$$\left(x,y,t,u\right)\rightarrow\left(\overline{x},\overline{y},\overline{t},\overline{u}\right),$$

we get

$$F\left(\overline{x},\overline{y},\overline{t},\overline{u},\overline{u_x},\overline{u_y},\overline{u_t},\overline{u_{xx}},\overline{u_{yy}},\overline{u_{tt}},\overline{u_{xy}},\overline{u_{xt}},\overline{u_{yt}}\right)=0. \tag{2.142}$$

2.2.2 Prolongation of infinitesimal generators

In this section we learn how to prolong infinitesimal generators involved in studying symmetries of PDEs. For the sake of simplicity, the derivation of prolongation formulas is restricted to symmetries of 2^{nd} order PDEs with one dependent variable u and three independent variables x,y,t. We consider the PDE

$$F\left(x,y,t,u,u_x,u_y,u_t,u_{xx},u_{yy},u_{tt},u_{xy},u_{xt},u_{yt}\right)=0, \tag{2.143}$$

with symmetry operator

$$X=\xi\left(x,y,t,u\right)\frac{\partial}{\partial x}+\theta\left(x,y,t,u\right)\frac{\partial}{\partial y}+\tau\left(x,y,t,u\right)\frac{\partial}{\partial t}+\phi\left(x,y,t,u\right)\frac{\partial}{\partial u}, \tag{2.144}$$

or equivalently the infinitesimal transformation

$$x^* = x+\epsilon\xi(x,y,t,u)+O\left(\epsilon^2\right), \tag{2.145}$$
$$y^* = y+\epsilon\theta(x,y,t,u)+O\left(\epsilon^2\right), \tag{2.146}$$
$$t^* = t+\epsilon\tau(x,y,t,u)+O\left(\epsilon^2\right), \tag{2.147}$$
$$u^* = u+\epsilon\phi(x,y,t,u)+O\left(\epsilon^2\right). \tag{2.148}$$

First Prolongation of X

In the first prolongation we find the transformation of the partial derivatives of 1^{st} order u_x,u_y and u_t.

i.e. we will find functions $\eta^{[x]}(x,y,t,u,u_x,u_y,u_t)$,$\eta^{[y]}(x,y,t,u,u_x,u_y,u_t)$ and $\eta^{[t]}(x,y,t,u,u_x,u_y,u_t)$ such that

$$u^*_{x^*} = u_x+\epsilon\eta^{[x]}(x,y,t,u,u_x,u_y,u_t)+O\left(\epsilon^2\right), \tag{2.149}$$
$$u^*_{y^*} = u_y+\epsilon\eta^{[y]}(x,y,t,u,u_x,u_y,u_t)+O\left(\epsilon^2\right), \tag{2.150}$$
$$u^*_{t^*} = u_t+\epsilon\eta^{[t]}(x,y,t,u,u_x,u_y,u_t)+O\left(\epsilon^2\right). \tag{2.151}$$

These functions, in terms of ξ,θ,τ and ϕ, are given by the following expressions

$$\eta^{[x]} = \phi_x+u_x\phi_u-u_x\xi_x-u_x^2\xi_u-u_y\theta_x-u_xu_y\theta_u-u_t\tau_x-u_xu_t\tau_u, \tag{2.152}$$
$$\eta^{[y]} = \phi_y+u_y\phi_u-u_x\xi_y-u_xu_y\xi_u-u_y\theta_y-u_y^2\theta_u-u_t\tau_y-u_yu_t\tau_u, \tag{2.153}$$
$$\eta^{[t]} = \phi_t+u_t\phi_u-u_x\xi_t-u_xu_t\xi_u-u_y\theta_t-u_yu_t\theta_u-u_t^2\tau_u-u_t\tau_t. \tag{2.154}$$

So we get the first prolongation of X

$$\begin{aligned}X^{[1]} &= \xi(x,y,t,u)\tfrac{\partial}{\partial x}+\theta(x,y,t,u)\tfrac{\partial}{\partial y}+\tau(x,y,t,u)\tfrac{\partial}{\partial t}+\phi(x,y,t,u)\tfrac{\partial}{\partial u}\\ &\quad+\eta^{[x]}\tfrac{\partial}{\partial u_x}+\eta^{[y]}\tfrac{\partial}{\partial u_y}+\eta^{[t]}\tfrac{\partial}{\partial u_t}.\end{aligned} \tag{2.155}$$

Second Prolongation of X

Analogous to the previous part, we want to find

$$X^{[2]} = X^{[1]}+\eta^{[xx]}\tfrac{\partial}{\partial u_{xx}}+\eta^{[yy]}\tfrac{\partial}{\partial u_{yy}}+\eta^{[tt]}\tfrac{\partial}{\partial u_{tt}}+\eta^{[xy]}\tfrac{\partial}{\partial u_{xy}}+\eta^{[xt]}\tfrac{\partial}{\partial u_{xt}}+\eta^{[yt]}\tfrac{\partial}{\partial u_{yt}}. \tag{2.156}$$

These functions are

$$\begin{aligned}\eta^{[xx]} &= -u_x^2u_y\theta_{uu}-u_x^2u_t\tau_{uu}-2u_xu_{xy}\theta_u-2u_{xy}\theta_x-2u_{xx}\xi_x+u_{xx}\phi_u\\ &-u_x^3\xi_{uu}+u_x^2\phi_{uu}+2u_x\phi_{xu}-u_x\xi_{xx}-2u_x^2\xi_{xu}-u_y\theta_{xx}-u_t\tau_{xx}-2u_xu_y\theta_{xu}\\ &-2u_xu_{xt}\tau_u-2u_{xt}\tau_x+\phi_{xx}-2u_xu_t\tau_{xu}-u_yu_{xx}\theta_u-3u_xu_{xx}\xi_u-u_tu_{xx}\tau_u,\end{aligned} \tag{2.157}$$

$$\begin{aligned}\eta^{[yy]} &= -u_x\xi_{yy}-2u_y^2\theta_{yu}-u_y\theta_{yy}+\phi_{yy}-u_t\tau_{yy}+u_y^2\phi_{uu}-u_y^3\theta_{uu}-2u_{xy}\xi_y\\ &+u_{yy}\phi_u-2u_{yy}\theta_y-2u_{yt}\tau_y-u_{yy}u_t\tau_u-u_xu_{yy}\xi_u-2u_xu_y\xi_{yu}-2u_yu_t\tau_{yu}\\ &-2u_yu_{yt}\tau_u-3u_yu_{yy}\theta_u-2u_yu_{xy}\xi_u-u_xu_y^2\xi_{uu}-u_y^2u_t\tau_{uu}+2u_y\phi_{yu},\end{aligned} \tag{2.158}$$

$$\begin{aligned}\eta^{[tt]} &= \phi_{tt}-u_xu_{tt}\xi_u-u_x\xi_{tt}-u_y\theta_{tt}-u_t\tau_{tt}-2u_t^2\tau_{tu}+u_t^2\phi_{uu}-u_t^3\tau_{uu}\\ &-2u_{xt}\xi_t-2u_{yt}\theta_t+u_{tt}\phi_u-2u_{tt}\tau_t-u_yu_t^2\theta_{uu}-2u_xu_t\xi_{tu}-u_xu_t^2\xi_{uu}\end{aligned}$$

$$-3u_tu_{tt}\tau_u-2u_tu_{xt}\xi_u-2u_yu_t\theta_{tu}-u_yu_{tt}\theta_u-2u_tu_{yt}\theta_u+2u_t\phi_{tu}, \tag{2.159}$$

$$\eta^{[xy]}=-2u_yu_{xy}\theta_u-u_tu_{xy}\tau_u-u_{xt}u_y\tau_u-u_xu_{yy}\theta_u-u_xu_{yt}\tau_u-u_xu_y\xi_{xu}$$
$$-u_{xy}\xi_x-u_{xx}u_y\xi_u-2u_xu_{xy}\xi_u+u_xu_y\phi_{uu}-u_x^2u_y\xi_{uu}-u_xu_y\theta_{yu}-u_xu_y^2\theta_{uu}$$
$$-u_xu_t\tau_{yu}-u_yu_t\tau_{xu}+u_{xy}\phi_u-u_{xy}\theta_y-u_{xt}\tau_y-u_{yy}\theta_x-u_{yt}\tau_x+u_x\phi_{yu}+\phi_{xy}$$
$$-u_xu_yu_t\tau_{uu}-u_x^2\xi_{yu}-u_{xx}\xi_y+u_y\phi_{xu}-u_x\xi_{xy}-u_y\theta_{xy}-u_y^2\theta_{xu}-u_t\tau_{xy}, \tag{2.160}$$

$$\eta^{[xt]}=-u_x\xi_{tx}-u_y\theta_{xt}-u_t\tau_{xt}-u_t^2\tau_{xu}+u_t\phi_{xu}+u_x\phi_{tu}-u_x^2\xi_{tu}-u_{xx}\xi_t$$
$$-u_{xy}\theta_t+u_{xt}\phi_u-u_{xt}\tau_t-u_{xt}\xi_x-u_{xy}\theta_x-u_{tt}\tau_x-u_xu_{tt}\tau_u-u_x^2u_t\xi_{uu}$$
$$-u_{xy}u_t\theta_u-u_xu_t\tau_{tu}-u_xu_y\theta_{tu}+u_xu_t\phi_{uu}-u_xu_t^2\tau_{uu}-u_{xx}u_t\xi_u-2u_xu_{xt}\xi_u$$
$$-u_{xt}u_y\theta_u-2u_{xt}u_t\tau_u-u_xu_t\xi_{xu}-u_yu_t\theta_{xu}-u_xu_{yt}\theta_u+\phi_{xt}-u_xu_yu_t\theta_{uu}, \tag{2.161}$$

$$\eta^{[yt]}=-u_xu_yu_t\xi_{uu}-u_y^2u_t\theta_{uu}-u_{xt}u_y\xi_u+u_yu_t\phi_{uu}-u_yu_t\tau_{tu}-u_yu_{tt}\tau_u$$
$$-u_xu_y\xi_{tu}-u_xu_t\xi_{yu}-u_{yy}u_t\theta_u-u_xu_{yt}\xi_u-u_yu_t\theta_{yu}-u_yu_t^2\tau_{uu}-2u_{yt}u_t\tau_u$$
$$-2u_yu_{yt}\theta_u-u_{xy}u_t\xi_u+\phi_{yt}-u_x\xi_{yt}-u_y\theta_{yt}-u_t\tau_{yt}-u_t^2\tau_{yu}+u_y\phi_{tu}+u_t\phi_{yu}$$
$$-u_y^2\theta_{tu}-u_{xy}\xi_t-u_{yy}\theta_t+u_{yt}\phi_u-u_{yt}\tau_t-u_{xt}\xi_y-u_{yt}\theta_y-u_{tt}\tau_y. \tag{2.162}$$

2.2.3 Definition of symmetries of PDEs using prolongation

Given the PDE

$$F\left(x,y,t,u,u_x,u_y,u_t,u_{xx},u_{yy},u_{tt},u_{xy},u_{xt},u_{yt}\right)=0, \tag{2.163}$$

A **symmetry** of PDE (2.163) is a one parameter group infinitesimal transformations:

$$x^*=x+\epsilon\xi\left(x,y,t,u\right)+O\left(\epsilon^2\right), \tag{2.164}$$

$$y^*=y+\epsilon\theta\left(x,y,t,u\right)+O\left(\epsilon^2\right), \tag{2.165}$$

$$t^*=t+\epsilon\tau\left(x,y,t,u\right)+O\left(\epsilon^2\right), \tag{2.166}$$

$$u^*=u+\epsilon\phi\left(x,y,t,u\right)+O\left(\epsilon^2\right), \tag{2.167}$$

or equivalently the infinitesimal generator

$$X=\xi\left(x,y,t,u\right)\frac{\partial}{\partial x}+\theta\left(x,y,t,u\right)\frac{\partial}{\partial y}+\tau\left(x,y,t,u\right)\frac{\partial}{\partial t}+\phi\left(x,y,t,u\right)\frac{\partial}{\partial u}, \tag{2.168}$$

that leaves the PDE (2.163) invariant, i.e.

$$F\left(\overline{x},\overline{y},\bar{t},\overline{u},\overline{u_x},\overline{u_y},\overline{u_t},\overline{u_{xx}},\overline{u_{yy}},\overline{u_{tt}},\overline{u_{xy}},\overline{u_{xt}},\overline{u_{yt}}\right)=0. \tag{2.169}$$

Treating equation (2.163) as an algebraic equation in $x,y,t,u,u_x,u_y,u_t,u_{xx},u_{yy},u_{tt},u_{xy},u_{xt},u_{yt}$, invariance criteria can be generalized to the following definition.

Definition:

The infinitesimal transformation

$$\overline{x}=f\left(x,y,t,u,\epsilon\right), \tag{2.170}$$

$$\overline{y}=g\left(x,y,t,u,\epsilon\right), \tag{2.171}$$

$$\bar{t} = h(x, y, t, u, \epsilon), \tag{2.172}$$

$$\bar{u} = j(x, y, t, u, \epsilon), \tag{2.173}$$

or equivalently the generator

$$X = \xi(x, y, t, u)\frac{\partial}{\partial x} + \theta(x, y, t, u)\frac{\partial}{\partial y} + \tau(x, y, t, u)\frac{\partial}{\partial t} + \phi(x, y, t, u)\frac{\partial}{\partial u}, \tag{2.174}$$

is a **symmetry** of PDE (2.163) if

$$X^{[2]}F|_{F=0} = 0, \tag{2.175}$$

where $X^{[2]}$ is the 2^{nd} prolongation of X.

2.2.4 Method of finding symmetries of PDEs

The equation

$$X^{[2]}F|_{F=0} = 0,$$

is called the **invariance criteria** for the symmetries of PDE (2.163). It is the basis for computing the symmetries of PDE (2.163). This will lead us to an over determined system of linear PDEs in ξ,θ,τ and ϕ. Solving this system will give us all possible functions ξ,θ,τ and ϕ that satisfy the invariants condition.

Steps for finding symmetries of (2.163)

1) Consider a symmetry generator of the form (2.168) and find the prolongation $X^{[2]}$.

2) Substitute the constraint $F = 0$ in $X^{[2]}F = 0$.

3) Obtain determining equations from step 2 by comparing coefficients of derivatives of u.

4) Simplify and solve the determining equations.

2.2.5 Reduction of PDEs using symmetries

Given a PDE

$$F(x, y, t, u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{tt}, u_{xy}, u_{xt}, u_{yt}) = 0, \tag{2.176}$$

with a symmetry

$$X = \xi(x, y, t, u)\frac{\partial}{\partial x} + \theta(x, y, t, u)\frac{\partial}{\partial y} + \tau(x, y, t, u)\frac{\partial}{\partial t} + \phi(x, y, t, u)\frac{\partial}{\partial u}. \tag{2.177}$$

The standard procedure, cf. [8,21], is to utilize the similarity variables to reduce the PDE (2.176) to an ODE where the similarity variables are the invariants of X obtained by solving $X(I) = 0$. The steps are similar to those in section 2.1.6.

CHAPTER 3
Minimal symmetries, group classification and reductions of nonlinear Klein Gordon equation on sphere

The aim of this chapter is to study the complete group classification problem and some symmetry reductions of the nonlinear Klein Gordon equation on the sphere given by

$$u_{tt} = u_{xx} + (\cot x)\, u_x + \left(\csc^2 x\right) u_{yy} + f\,(u)\,, \tag{3.1}$$

with $f\,(u)$ is an arbitrary nonlinear function.

The group classification of equation (3.1) is carried out in section 3.1. It is based on necessary conditions on $f(u)$ which are obtained through a triangulation of determining equations of Lie symmetries of equation (3.1).

An efficient method to obtain such triangulation is the well-known method of Mansfield [29] of generating differential Grobner bases of determining equations. A variant of Mansfield's method will be used to generate cases of $f(u)$ and hence for carrying out the group classification. Precisely, the following result will be proved.

Theorem 3.1:

The Lie symmetry algebra of the nonlinear PDE (3.1) for all $f\,(u)$ is four dimensional and is generated by

$$X_1 = \cos y \tfrac{\partial}{\partial x} - \sin y \cot x \tfrac{\partial}{\partial y},$$
$$X_2 = \sin y \tfrac{\partial}{\partial x} + \cos y \cot x \tfrac{\partial}{\partial y},$$
$$X_3 = \tfrac{\partial}{\partial t},$$
$$X_4 = \tfrac{\partial}{\partial y},$$

and there are no nonlinear functions $f\,(u)$ for which the PDE (3.1) admits symmetries more than the minimal symmetries .

The proof of theorem 3.1 is contained in section 3.1. Examples of reductions and some exact solutions, corresponding to the cases obtained in section 3.1, are provided in section 3.2.

3.1 Minimal symmetry algebra and group classification

We are going to obtain the Lie symmetries of the equation (3.1). To do so, let

$$X = \xi(x, y, t, u) \frac{\partial}{\partial x} + \theta(x, y, t, u) \frac{\partial}{\partial y} + \tau(x, y, t, u) \frac{\partial}{\partial t} + \phi(x, y, t, u) \frac{\partial}{\partial u}, \tag{3.2}$$

be symmetry of the PDE (3.1).

If $X^{[2]}$ denotes the second prolongation of X then the invariance condition for finding ξ, θ, τ and ϕ is given by

$$X^{[2]} \left(u_{tt} - \cot x u_x - u_{xx} - \csc^2 x u_{yy} - f(u)\right) = 0, \tag{3.3}$$

where

$$\begin{aligned} X^{[2]} = {} & X + \eta^{[x]} \frac{\partial}{\partial u_x} + \eta^{[y]} \frac{\partial}{\partial u_y} + \eta^{[t]} \frac{\partial}{\partial u_t} + \eta^{[xx]} \frac{\partial}{\partial u_{xx}} + \eta^{[yy]} \frac{\partial}{\partial u_{yy}} + \eta^{[tt]} \frac{\partial}{\partial u_{tt}} \\ & + \eta^{[xy]} \frac{\partial}{\partial u_{xy}} + \eta^{[xt]} \frac{\partial}{\partial u_{xt}} + \eta^{[yt]} \frac{\partial}{\partial u_{yt}}. \end{aligned} \tag{3.4}$$

Applying $X^{[2]}$ to equation (3.1), will lead to

$$\begin{aligned} & -\phi f_u - \phi_{xx} + \phi_{tt} - u_y \theta_{tt} - u_t \tau_{tt} + u_t^2 \phi_{uu} - u_t^3 \tau_{uu} - 2u_{xt} \xi_t - 2u_{yt} \theta_t \\ & - \csc^2 x \phi_{yy} - 2\csc^2 x u_y \phi_{yu} + \csc^2 x u_y^2 u_t \tau_{uu} + \csc^2 x u_y \theta_{yy} + 2\csc^2 x u_y^2 \theta_{yu} \\ & + \csc^2 x u_t \tau_{yy} - \csc^2 x u_y^2 \phi_{uu} + \csc^2 x u_y^3 \theta_{uu} + 2\csc^2 x u_{xy} \xi_y - \csc^2 x u_{yy} \phi_u \\ & + 2\csc^2 x u_{yy} \theta_y + 2\csc^2 x u_{yt} \tau_y + 2\csc^2 x u_{xy} u_y \xi_u + 2\csc^2 x u_y u_t \tau_{yu} \\ & + 3\csc^2 x u_y u_{yy} \theta_u + \csc^2 x u_{yy} u_t \tau_u + 2\csc^2 x u_y u_{yt} \tau_u + u_{xx} \phi_u + \csc^2 x u_{yy} \phi_u \\ & + f\phi_u - 2u_{xx} \tau_t - 2\csc^2 x u_{yy} \tau_t - 2f\tau_t - u_{xx} u_y \theta_u - \csc^2 x u_y u_{yy} \theta_u - f u_y \theta_u \\ & - 3u_{xx} u_t \tau_u - 3\csc^2 x u_{yy} u_t \tau_u - 3f u_t \tau_u - \cot x \phi_x + \cot x u_t \tau_x + \cot x u_y \theta_x \\ & + 2u_x^2 \xi_{xu} + u_x^2 u_t \tau_{uu} + u_x^2 u_y \theta_{uu} - u_x^2 \phi_{uu} + 2u_x u_{xy} \theta_u - 2u_x \phi_{xu} + \cot x u_x \phi_u \\ & - 2u_x u_t \xi_{tu} + 3u_x u_{xx} \xi_u - \cot x u_x u_y \theta_u - 3\cot x u_x u_t \tau_u - 2\cot x u_x \tau_t \\ & + \csc^2 x u_x \xi - u_x u_t^2 \xi_{uu} + 2u_x u_y \theta_{xu} - u_x \xi_{tt} + u_x \xi_{xx} + 2u_x u_t \tau_{xu} + 2u_x u_{xt} \tau_u \\ & + \csc^2 x u_x u_{yy} \xi_u + \csc^2 x u_x u_y^2 \xi_{uu} + \csc^2 x u_x \xi_{yy} + 2\csc^2 x u_x u_y \xi_{yu} - \cot x u_x \phi_u \\ & + \cot x u_x u_t \tau_u + \cot x u_x u_y \theta_u + \cot x u_x \xi_x - u_x u_{xx} \xi_u - \csc^2 x u_x u_{yy} \xi_u - f u_x \xi_u \\ & + u_{xx} u_t \tau_u + u_{xx} u_y \theta_u + 2\cot x \csc^2 x u_{yy} \xi - 2u_t^2 \tau_{tu} - 2u_y u_t \theta_{tu} + u_y \theta_{xx} + u_t \tau_{xx} \\ & - u_{xx} \phi_u + 2u_{xx} \xi_x + 2u_{xy} \theta_x + 2u_{xt} \tau_x + u_x^3 \xi_{uu} - 2u_t u_{xt} \xi_u - u_y u_t^2 \theta_{uu} \\ & - 2u_t u_{yt} \theta_u + 2u_t \phi_{tu} = 0. \end{aligned} \tag{3.5}$$

Now, we compare the coefficients of different derivatives of u to get the determining equations

Constant Term : $-\phi f_u - \phi_{xx} + \phi_{tt} - \csc^2 x \phi_{yy} + f\phi_u - 2f\tau_t - \cot x \phi_x = 0,$

Coefficients of u_x : $-2\phi_{xu} + \cot x \phi_u - 2\cot x \tau_t + \csc^2 x \xi - \xi_{tt} + \xi_{xx}$
$+ \csc^2 x \xi_{yy} - \cot x \phi_u + \cot x \xi_x - f\xi_u = 0,$

Coefficients of u_y : $-\theta_{tt} + \theta_{xx} - 2\csc^2 x \xi_{yy} \phi_{yu} + \csc^2 x \theta_{yy} - f\theta_u + \cot x \theta_x = 0,$

Coefficients of u_t : $-\tau_{tt} + \tau_{xx} + \csc^2 x \tau_{yy} - 3f\tau_u + \cot x \tau_x + 2\phi_{tu} = 0,$

Coefficients of $u_x^2 \quad : 2\xi_{xu} - \phi_{uu} = 0,$

Coefficients of $u_y u_{xy} : 2\csc^2 x \xi_u = 0,$

Coefficients of $u_x u_{xy} : \theta_u = 0,$

Coefficients of $u_x u_{xt} : \tau_u = 0,$

Coefficients of $u_x u_{xx} : 3\xi_u - \xi_u = 0,$

Coefficients of $u_{xx} \quad : \phi_u - 2\tau_t - \phi_u + 2\xi_x = 0,$

Coefficients of $u_{xt} \quad : 2\tau_x - 2\xi_t = 0,$

Coefficients of $u_{xy} \quad : 2\csc^2 x \xi_y + 2\theta_x = 0,$

Coefficients of $u_{yt} \quad : 2\csc^2 x \tau_y - 2\theta_t = 0,$

Coefficients of $u_{yy} \quad : \cot x \xi - \tau_t + \theta_y = 0.$

The above system can be simplified to get the following system of determining equations with the order

$$\phi > \tau > \theta > \xi > f, \text{ and } x > y > t > u,$$

$$e_1 : \xi_u = 0,$$

$$e_2 : \theta_u = 0,$$

$$e_3 : \csc^2 x \xi_y + \theta_x = 0,$$

$$e_4 : \tau_u = 0,$$

$$e_5 : \xi_x - \tau_t = 0,$$

$$e_6 : \cot x \xi - \tau_t + \theta_y = 0,$$

$$e_7 : \csc^2 x \tau_y - \theta_t = 0,$$

$$e_8 : \tau_x - \xi_t = 0,$$

$$e_9 : \phi_{uu} = 0,$$

$$e_{10} : \cot x \tau_x - \tau_{tt} + \tau_{xx} + \csc^2 x \tau_{yy} + 2\phi_{tu} = 0,$$

$$e_{11} : \cot x \theta_x - \theta_{tt} + \theta_{xx} + \csc^2 x \theta_{yy} - 2\csc^2 x \phi_{yu} = 0,$$

$$e_{12} : -\cot x \xi_x - \xi_{tt} + \xi_{xx} + \csc^2 x \xi + \csc^2 x \xi_{yy} - 2\phi_{xu} = 0,$$

$$e_{13} : f\phi_u - \phi f_u - \phi_{xx} + \phi_{tt} - 2f\tau_t - \csc^2 x \phi_{yy} - \cot x \phi_x = 0.$$

The system of determining equations will be solved utilizing a triangulation procedure based on techniques for obtaining differential Groebner basis developed by Mansfield in [29].

The operation $(e_5)_x - (e_8)_t$ gives

$$\xi_{xx} - \tau_{xt} + \tau_{xt} - \xi_{tt} = 0,$$

$$\Longrightarrow \xi_{xx} - \xi_{tt} = 0. \tag{3.6}$$

$$\csc^2 x \, (e_5)_y + (e_7)_t = 0,$$

$$\Longrightarrow \csc^2 x \xi_{xy} - \csc^2 x \tau_{ty} + \csc^2 x \tau_{ty} - \theta_{tt} = 0,$$

$$\Longrightarrow \csc^2 x\tau_{yt} - \theta_{tt} = 0. \tag{3.7}$$

$$(e_5)_x - (e_8)_t = 0 \Longrightarrow \xi_{xt} - \tau_{tt} + \tau_{xx} - \xi_{xt} = 0,$$

$$\Longrightarrow \tau_{xx} - \tau_{tt} = 0. \tag{3.8}$$

$$\csc^2 x\,(e_6)_y + (e_7)_t = 0,$$

$$\Longrightarrow \csc^2 x \cot x\xi_y - \csc^2 x\tau_{ty} + \csc^2 x\theta_{yy} + \csc^2 x\tau_{ty} - \theta_{tt} = 0,$$

$$\Longrightarrow \csc^2 x \cot x\xi_y + \csc^2 x\theta_{yy} - \theta_{tt} = 0. \tag{3.9}$$

$$(e_6)_x + (e_8)_t = 0,$$

$$\Longrightarrow \cot x\xi_x - \csc^2 x\xi - \tau_{xt} + \theta_{xy} + \tau_{xt} - \xi_{tt} = 0,$$

$$\Longrightarrow \cot x\xi_x - \csc^2 x\xi + \theta_{xy} - \xi_{tt} = 0. \tag{3.10}$$

$$(e_7)_x - \csc^2 x\,(e_8)_y = 0,$$

$$\Longrightarrow -2\cot x \csc^2 x\tau_y + \csc^2 x\tau_{xy} - \theta_{xt} - \csc^2 x\tau_{xy} + \csc^2 x\xi_{yt} = 0,$$

$$\Longrightarrow -2\cot x \csc^2 x\tau_y + \csc^2 x\xi_{yt} - \theta_{xt} = 0. \tag{3.11}$$

$$(e_{13})_u = 0 \Longrightarrow -f_{uu}\phi - \phi_{xxu} + \phi_{ttu} - 2f_u\tau_t - \csc^2 x\phi_{yyu} - \cot x\phi_{xu} = 0, \tag{3.12}$$

$$(3.12)_u = 0 \Longrightarrow f_{uu}\phi_u + f_{uuu}\phi + 2f_{uu}\tau_t = 0, \tag{3.13}$$

$$(3.13)_u = 0 \Longrightarrow 2f_{uuu}\phi_u + f_{uuuu}\phi + 2f_{uuu}\tau_t = 0, \tag{3.14}$$

$$(3.14)_u = 0 \Longrightarrow 3f_{uuuu}\phi_u + f_{uuuuu}\phi + 2f_{uuuu}\tau_t = 0. \tag{3.15}$$

$$f_{uu}\,(3.14) - 2f_{uuu}\,(3.13) = 0,$$

$$\Longrightarrow 2f_{uu}f_{uuu}\phi_u + f_{uu}f_{uuuu}\phi + 2f_{uu}f_{uuu}\tau_t - 2f_{uu}f_{uuu}\phi_u - 2f_{uuu}^2\phi$$
$$-4f_{uu}f_{uuu}\tau_t = 0,$$

$$\Longrightarrow \left(f_{uu}f_{uuuu} - 2f_{uuu}^2\right)\phi - (2f_{uu}f_{uuu})\,\tau_t = 0, \tag{3.16}$$

$$(3.16)_u = 0 \Longrightarrow$$

$$\left(f_{uu}f_{uuuu} - 2f_{uuu}^2\right)\phi_u + \left(f_{uu}f_{uuuuu} - 3f_{uuu}f_{uuuu}\right)\phi - 2\left(f_{uuu}^2 + f_{uu}f_{uuuu}\right)\tau_t = 0. \tag{3.17}$$

Use (3.13) and (3.17) to eliminate ϕ_u,

$$f_{uu}\,(3.17) - \left(f_{uu}f_{uuuu} - 2f_{uuu}^2\right)(3.13) = 0,$$

$$\Longrightarrow \left(f_{uu}^2 f_{uuuuu} - 3f_{uu}f_{uuu}f_{uuuu} - f_{uu}f_{uuu}f_{uuuu} + 2f_{uuu}^3\right)\phi$$
$$-2\left(f_{uu}f_{uuu}^2 + f_{uu}^2 f_{uuuu} + f_{uu}^2 f_{uuuu} - 2f_{uu}f_{uuu}^2\right)\tau_t = 0,$$

$$\Longrightarrow \left(f_{uu}^2 f_{uuuuu} - 4f_{uu}f_{uuu}f_{uuuu} + 2f_{uuu}^3\right)\phi + \left(-4f_{uu}^2 f_{uuuu} + 2f_{uu}f_{uuu}^2\right)\tau_t = 0. \tag{3.18}$$

Use (3.16) and (3.18) to eliminate ϕ,

$$\left(f_{uu}f_{uuuu} - 2f_{uuu}^2\right)(3.18) - \left(f_{uu}^2 f_{uuuuu} - 4f_{uu}f_{uuu}f_{uuuu} + 2f_{uuu}^3\right)(3.16) = 0.$$

Then, the coefficient of τ_t will be

$$-4f_{uu}^3 f_{uuuu}^2 + 2f_{uu}^2 f_{uuu}^2 f_{uuuu} + 8f_{uu}^2 f_{uuu}^2 f_{uuuu} - 4f_{uu}f_{uuu}^4 + 2f_{uu}^3 f_{uuu}f_{uuuuu}$$
$$-8f_{uu}^2 f_{uuu}^2 f_{uuuu} + 4f_{uu}f_{uuu}^4,$$

$$\Longrightarrow \left(f_{uu}f_{uuu}f_{uuuuu} - 2f_{uu}f_{uuuu}^2 + f_{uuu}^2 f_{uuuu}\right)\tau_t = 0. \tag{3.19}$$

If $\tau_t = 0$, then equation (3.19) is satisfied for any function $f(u)$.

$$(e_7)_y = 0 \Longrightarrow \csc^2 x \tau_{yy} = \theta_{yt}, \tag{3.20}$$

$$(e_6)_t = 0 \Longrightarrow \cot x \xi_t - \tau_{tt} + \theta_{ty} = 0 \Longrightarrow \cot x \tau_x - \tau_{tt} + \theta_{ty} = 0,$$

$$\Longrightarrow \cot x \tau_x - \tau_{tt} + \csc^2 x \tau_{yy} = 0. \tag{3.21}$$

Plug in equation (e_{10}), get

$$\tau_{xx} + 2\phi_{tu} = 0,$$

$$\Longrightarrow \tau_{tt} + 2\phi_{tu} = 0. \tag{3.22}$$

From equation $(e_3) \Longrightarrow \csc^2 x \xi_y = -\theta_x$,

Equation (3.9) $\Longrightarrow \cot x \csc^2 x \xi_y + \csc^2 x \theta_{yy} - \theta_{tt} = 0.$ (3.23)

Plug in equation (e_{11}), get

$$-2 \cot x \csc^2 x \xi_y + \theta_{xx} - 2 \csc^2 x \phi_{yu} = 0. \tag{3.24}$$

Also

$$(e_3)_x = 0 \Longrightarrow -2 \cot x \csc^2 x \xi_y + \csc^2 x \xi_{xy} + \theta_{xx} = 0. \tag{3.25}$$

Plug equation (3.25) in equation (3.24), get

$$\csc^2 x \xi_{xy} - 2 \csc^2 x \phi_{yu} = 0,$$

$$\Longrightarrow \tau_{ty} - 2\phi_{yu} = 0. \tag{3.26}$$

Now

$$(e_3)_y - (e_6)_x = 0 \Longrightarrow \theta_{xy} + \csc^2 x \xi_{yy} + \csc^2 x \xi - \cot x \xi_x + \tau_{xt} - \theta_{xy} = 0,$$

$$\Longrightarrow \csc^2 x \xi_{yy} + \csc^2 x \xi - \cot x \xi_x + \tau_{xt} = 0. \tag{3.27}$$

Plug in equation (e_{12}), get

$$\tau_{xt} + 2\phi_{xu} = 0. \tag{3.28}$$

Now

Since $\tau_t = 0$, from equation (e_5) we have

$$\xi_x = 0,$$

$$\Longrightarrow \tau_{xx} = 0, \qquad \text{by equation (3.8)} \tag{3.29}$$

$$\Longrightarrow \xi_{tt} = 0, \qquad \text{by equation (3.6)} \tag{3.30}$$

By equation (e_{24}), and the nonlinearity of $f(u)$ we have

$$\phi = 0, \tag{3.31}$$

$\xi_x = 0$, and $\xi_u = 0$ implies that

$$\xi = \xi(y, t), \tag{3.32}$$

$\xi_{tt} = 0$, implies that ξ is linear in t,

$$\xi = A(y) t + B(y). \tag{3.33}$$

And

$\tau_t = 0$, and $\tau_u = 0$ implies that

$$\tau = \tau(x, y), \tag{3.34}$$

$\tau_{xx} = 0$, implies that τ is linear in x,

$$\tau = C(y)\, x + D(y), \tag{3.35}$$

$$\tau_x = \xi_t \Longrightarrow A(y) = C(y). \tag{3.36}$$

Now, we try to determine $A(y), B(y)$ and $D(y)$,

Since $\phi = 0$, equation (e_{10}) becomes

$$\cot x\tau_x + \csc^2 x\tau_{yy} = 0 \Longrightarrow \cot x A(y) + \csc^2 x\left[A^{''}(y)\, x + D^{''}(y)\right] = 0, \tag{3.37}$$

This is possible only if

$$A(y) = 0, \tag{3.38}$$

and

$$D^{''}(y) = 0 \Longrightarrow D(y) = ay + b. \tag{3.39}$$

And equation (e_{12}) becomes

$$\csc^2 x\xi_{yy} + \csc^2 x\xi = 0 \Longrightarrow \xi_{yy} + \xi = 0 \Longrightarrow B^{''}(y) + B(y) = 0,$$

$$\Longrightarrow B(y) = k_1 \cos y + k_2 \sin y. \tag{3.40}$$

So, we have

$$\phi = 0, \tag{3.41}$$

$$\xi = k_1 \cos y + k_2 \sin y, \tag{3.42}$$

$$\tau = ay + b. \tag{3.43}$$

Next, we note that

$$\theta_u = 0 \Longrightarrow \theta = \theta(x, y, t),$$

$$(e_7)_t = 0 \Longrightarrow \csc^2 x\tau_{yt} - \theta_{tt} = 0,$$

$$\Longrightarrow \theta_{tt} = 0, \tag{3.44}$$

which implies that θ is linear in t,

$$\Longrightarrow \theta = E(x, y)\, t + F(x, y), \tag{3.45}$$

$$(e_6)_t = 0 \Longrightarrow \cot x\xi_t - \tau_{tt} + \theta_{ty} = 0 \Longrightarrow \theta_{ty} = 0 \Longrightarrow E_y = 0 \Longrightarrow E = E(x),$$

$$\Longrightarrow \theta = E(x)\, t + F(x, y), \tag{3.46}$$

$$(e_3)_t = 0 \Longrightarrow \csc^2 x\xi_{yt} + \theta_{xt} = 0 \Longrightarrow \theta_{xt} = 0 \Longrightarrow E_x = 0 \Longrightarrow E = c.$$

To determine a and c, we note that equation (e_7) becomes

$$a \csc^2 x - c = 0 \Longrightarrow a = c = 0. \tag{3.47}$$

To determine $F(x, y)$, we note that

$$(e_3) \Longrightarrow F_x = -\csc^2 x\left[-k_1 \sin y + k_2 \cos y\right], \tag{3.48}$$

$$(e_6) \Longrightarrow F_y = -\cot x\left[k_1 \cos y + k_2 \sin y\right], \tag{3.49}$$

Integrating equation (3.48), we get

$$F = \cot x\left[-k_1 \sin y + k_2 \cos y\right] + g\left(y\right), \tag{3.50}$$

Using equation (3.50) in equation (3.49), we get

$$-\cot x\left[k_1 \cos y + k_2 \sin y\right] + g^{'}\left(y\right) = -\cot x\left[k_1 \cos y + k_2 \sin y\right],$$

$$\Longrightarrow g^{'}\left(y\right) = 0 \Longrightarrow g\left(y\right) = c, \tag{3.51}$$

$$\Longrightarrow F\left(x, y\right) = \cot x\left[-k_1 \sin y + k_2 \cos y\right] + c. \tag{3.52}$$

So, we have

$$\phi = 0, \tag{3.53}$$

$$\xi = k_1 \cos y + k_2 \sin y, \tag{3.54}$$

$$\tau = k_3, \tag{3.55}$$

$$\theta = \cot x\left[-k_1 \sin y + k_2 \cos y\right] + k_4. \tag{3.56}$$

Now:

Putting $k_1 = 1$ and the remaining constants vanish implies

$$X_1 = \cos y \tfrac{\partial}{\partial x} - \sin y \cot x \tfrac{\partial}{\partial y}. \tag{3.57}$$

Putting $k_2 = 1$ and the remaining constants vanish implies

$$X_2 = \sin y \tfrac{\partial}{\partial x} + \cos y \cot x \tfrac{\partial}{\partial y}. \tag{3.58}$$

Putting $k_3 = 1$ and the remaining constants vanish implies

$$X_3 = \tfrac{\partial}{\partial t}. \tag{3.59}$$

Putting $k_4 = 1$ and the remaining constants vanish implies

$$X_4 = \tfrac{\partial}{\partial y}. \tag{3.60}$$

So: the **minimal** algebra is generated by

$$X_1 = \cos y \tfrac{\partial}{\partial x} - \sin y \cot x \tfrac{\partial}{\partial y},$$

$$X_2 = \sin y \tfrac{\partial}{\partial x} + \cos y \cot x \tfrac{\partial}{\partial y},$$

$$X_3 = \tfrac{\partial}{\partial t},$$

$$X_4 = \tfrac{\partial}{\partial y},$$

with commutation relations given in the table below

Table 3

Commutator table for the Lie algebra

	X_1	X_2	X_3	X_4
X_1	0	$-X_4$	0	X_2
X_2	X_4	0	0	$-X_1$
X_3	0	0	0	0
X_4	$-X_2$	X_1	0	0

.

To look for functions $f(u)$ that may give larger symmetry algebra we assume that $\tau_t \neq 0$ and solve the differential equation (3.19).

Let us consider the equation

$$f_{uu}f_{uuu}f_{uuuuu}-2f_{uu}f_{uuuu}^2+f_{uuu}^2f_{uuuu}=0, \tag{3.61}$$

where $f(u)$ is nonlinear

Let $H=f_{uu}$, then equation (3.61) becomes

$$HH_uH_{uuu}-2HH_{uu}^2+H_u^2H_{uu}=0, \tag{3.62}$$

which implies that

Either

$$H_{uu}=0 \Longrightarrow H_u=c_1,$$

If $c_1=0 \Longrightarrow H=c,$

$$\Longrightarrow f(u)=au^2+bu+c, \tag{3.63}$$

And if $c_1 \neq 0 \Longrightarrow H=c_1u+c_2 \Longrightarrow f_u=c_1u^2+c_2u+c_3,$

$$\Longrightarrow f(u)=au^3+bu^2+cu+e. \tag{3.64}$$

Or

$$H_{uu}\neq 0,$$

So we can divide equation (3.62) by HH_uH_{uu}, to get

$$\tfrac{H_{uuu}}{H_{uu}}-2\tfrac{H_{uu}}{H_u}+\tfrac{H_u}{H}=0. \tag{3.65}$$

Integrating equation (3.65) we get

$$\Longrightarrow \ln H_{uu}-\ln H_u^2+\ln H=c_0 \Longrightarrow \ln\tfrac{HH_{uu}}{H_u^2}=c_0 \Longrightarrow \tfrac{HH_{uu}}{H_u^2}=c,$$

$$\Longrightarrow HH_{uu}-cH_u^2=0. \tag{3.66}$$

Divide equation (3.66) by HH_u, we get

$$\tfrac{H_{uu}}{H_u}-c\tfrac{H_u}{H}=0, \tag{3.67}$$

Integrating equation (3.67) we get

$$\Longrightarrow \ln H_u-c\ln H=k_0 \Longrightarrow \ln\tfrac{H_u}{H^c}=k_0,$$

$$\Longrightarrow \frac{H_u}{H^c} = k, \tag{3.68}$$

If $c = 1$, then equation (3.68) becomes

$$\frac{H_u}{H} = k, \tag{3.69}$$

Integrating equation (3.69) we get

$$\Longrightarrow \ln H = ku + k_1 \Longrightarrow H = k_2 e^{ku} \Longrightarrow f_{uu} = k_2 e^{ku} \Longrightarrow f_u = k_3 e^{ku} + k_4,$$

$$\Longrightarrow f(u) = k_3 e^{ku} + k_4 u + k_5, \tag{3.70}$$

And if $c \neq 1$, then equation (3.68) becomes

$$H^{-c}\frac{dH}{du} = k \Longrightarrow H^{-c}dH = kdu \Longrightarrow \frac{1}{1-c}H^{1-c} = ku + k_1 \Longrightarrow H^{1-c} = k_2 u + k_3,$$

$$\Longrightarrow \left(H^{1-c}\right)^{\frac{1}{1-c}} = (k_2 u + k_3)^{\frac{1}{1-c}} \Longrightarrow H = k_4 (u + k_5)^n,$$

$$\Longrightarrow f_{uu} = k_4 (u + k_5)^n, \quad \text{where } n = \frac{1}{1-c} \text{ and } c \neq 1. \tag{3.71}$$

Equation (3.71) can be viewed into three different cases:

Case1: $n \neq -1, -2$,

$$\Longrightarrow f_u = \frac{k_4}{n+1}(u + k_5)^{n+1} + k_6 \Longrightarrow f = k_7 (u + k_5)^{n+2} + k_6 u + k_8,$$

$$\Longrightarrow f = k_7 (u + k_5)^m + k_6 u + k_8, \text{ where } m \neq 0, 1. \tag{3.72}$$

Case2: $n = -2$,

$$\Longrightarrow f_u = -k_4 (u + k_5)^{-1} + k_6 \Longrightarrow f = -k_4 \ln(u + k_5) + k_6 u + k_8, \tag{3.73}$$

Case3: $n = -1$,

$$\Longrightarrow f_{uu} = k_4 (u + k_5)^{-1} \Longrightarrow f_u = k_4 \ln(u + k_5) + k_6,$$

$$\Longrightarrow f = k_4 ((u + k_5)\ln(u + k_5) - (u + k_5)) + k_9 u + k_{10}. \tag{3.74}$$

In summary, solving equation (3.61) gives six different nonlinear functions:

1. $f(u)$ is quadratic, i.e. $f(u) = u^2 + au + b$,
2. $f(u)$ is cubic, i.e. $f(u) = u^3 + au^2 + bu + c$,
3. $f(u) = ae^{bu} + cu + d$, where $a, b \neq 0$,
4. $f(u) = (u + b)^m + cu + d$, where $m \neq 0, 1, 2, 3$,
5. $f(u) = \ln(u + b) + cu + d$,
6. $f(u) = [(u + b)\ln(u + b) - (u + b)] + cu + d$,

To figure out if any function will give extra symmetries, we establish the following equations:

Use equations (3.13) and (3.17) to eliminate ϕ,

$$f_{uuu}(3.17) - (f_{uu}f_{uuuuu} - 3f_{uuu}f_{uuuu})(3.13) = 0,$$

$$\Longrightarrow \left(f_{uu}f_{uuu}f_{uuuu} - 2f_{uuu}^3 - f_{uu}^2 f_{uuuuu} + 3f_{uu}f_{uuu}f_{uuuu}\right)\phi_u$$

$$-2\left(f_{uuu}^3 + f_{uu}f_{uuu}f_{uuuu} - f_{uu}^2 f_{uuuuu} + 3f_{uu}f_{uuu}f_{uuuu}\right)\tau_t = 0,$$

$$\Longrightarrow \left(-2f_{uuu}^3 - f_{uu}^2 f_{uuuuu} + 4f_{uu}f_{uuu}f_{uuuu}\right)\phi_u + \left(-2f_{uuu}^3 - 2f_{uu}^2 f_{uuuuu} + 4f_{uu}f_{uuu}f_{uuuu}\right)\tau_t = 0,$$

$$\Longrightarrow \phi_u = \left(\frac{2f_{uuu}^3+2f_{uu}^2f_{uuuuu}-4f_{uu}f_{uuu}f_{uuuu}}{-2f_{uuu}^3-f_{uu}^2f_{uuuuu}+4f_{uu}f_{uuu}f_{uuuu}}\right)\tau_t. \tag{3.75}$$

From equation (3.13) we have

$$\phi = \frac{-f_{uu}}{f_{uuu}}\left(\phi_u + 2\tau_t\right). \tag{3.76}$$

The analysis for extra symmetries, if any, for different forms of $f(u)$ are summarized in the following cases:

Case1:

$f(u)$ is quadratic, i.e. $f(u) = u^2 + au + b$, then

$f_u = 2u + a,$

$f_{uu} = 2,$

$f_{uuu} = 0,$

$f_{uuuu} = 0.$

From equation (3.13), we get

$$2\phi_u + 4\tau_t = 0 \Longrightarrow \phi_u = -2\tau_t \Longrightarrow \tau_t = -\tfrac{1}{2}\phi_u, \tag{3.77}$$

From equation (3.12) we get

$$-2\phi - 2\left(2u + a\right)\tau_t = 0 \Longrightarrow \phi = -\left(2u + a\right)\tau_t. \tag{3.78}$$

Also

$$\phi_{xu} = -2\tau_{xt} = -2\xi_{xx}, \tag{3.79}$$

$$\phi_{yu} = -2\tau_{yt} = -2\xi_{xy}, \tag{3.80}$$

$$\phi_{tu} = -2\tau_{tt} = -2\xi_{xt}. \tag{3.81}$$

Now

$$(3.22) \Longrightarrow \tau_{tt} + 2\phi_{tu} = 0 \Longrightarrow \tau_{tt} - 4\tau_{tt} = 0 \Longrightarrow \tau_{tt} = 0 \Longrightarrow \xi_{xt} = 0, \tag{3.82}$$

$$(3.26) \Longrightarrow \tau_{yt} - 2\phi_{yu} = 0 \Longrightarrow \tau_{yt} + 4\tau_{yt} = 0 \Longrightarrow \tau_{yt} = 0 \Longrightarrow \xi_{xy} = 0, \tag{3.83}$$

$$(3.28) \Longrightarrow \tau_{xt} + 2\phi_{xu} = 0 \Longrightarrow \tau_{xt} - 4\tau_{xt} = 0 \Longrightarrow \tau_{xt} = 0 \Longrightarrow \xi_{xx} = 0. \tag{3.84}$$

Also

$\phi_{uu} = 0$, implies that ϕ is linear in u i.e.

$$\phi = A\left(x,y,t\right)u + B\left(x,y,t\right), \tag{3.85}$$

$$\Longrightarrow \phi_u = A\left(x,y,t\right),$$

$$\Longrightarrow \tau_t = -\tfrac{1}{2}A\left(x,y,t\right). \tag{3.86}$$

since $\tau_{tt} = 0 \Longrightarrow A = A\left(x,y\right),$

$\tau_{yt} = 0 \Longrightarrow A = A\left(x\right),$

$$\tau_{xt} = 0 \Longrightarrow A = c, \tag{3.87}$$

$$\Longrightarrow \phi = \tfrac{1}{2}A\left(2u + a\right), \tag{3.88}$$

$$\Longrightarrow \phi_u = A. \tag{3.89}$$

Plug in equation (e_{13}),

$$\Longrightarrow \left(u^2 + au + b\right) A - \tfrac{1}{2} A \left(2u + a\right)\left(2u + a\right) + \left(u^2 + au + b\right) A = 0,$$

$$\Longrightarrow Au^2 + aAu + bA - \tfrac{1}{2} A \left(4u^2 + 4au + a^2\right) + Au^2 + aAu + bA = 0,$$

$$\Longrightarrow 2Au^2 + 2aAu + 2bA - 2Au^2 - 2aAu - \tfrac{1}{2} a^2 A = 0,$$

$$\Longrightarrow 2bA - \tfrac{1}{2} a^2 A = 0. \tag{3.90}$$

If $A = 0$, then $\tau_t = 0$, which gives the minimal symmetry algebra.

and if $A \neq 0$, then

$$\Longrightarrow 2b = \tfrac{1}{2} a^2 \Longrightarrow b = \tfrac{1}{4} a^2. \tag{3.91}$$

So, we must have

$$f\left(u\right) = u^2 + au + \tfrac{1}{4} a^2,$$

$$\Longrightarrow f\left(u\right) = \left(u + \tfrac{1}{2} a\right)^2, \tag{3.92}$$

$\Longrightarrow f\left(u\right)$ is a perfect square

So far we have

$$\xi_x = -\tfrac{1}{2} A, \xi_u = 0, \xi_{xx} = 0, \xi_{tt} = 0 \Longrightarrow \xi = f\left(y, t\right) x + g\left(y, t\right) \Longrightarrow \xi = -\tfrac{1}{2} Ax + g\left(y, t\right),$$

$$\xi_t = g_t \Longrightarrow g_{tt} = 0 \Longrightarrow g\left(y, t\right) = a\left(y\right) t + b\left(y\right),$$

$$\Longrightarrow \xi = -\tfrac{1}{2} Ax + a\left(y\right) t + b\left(y\right). \tag{3.93}$$

Also

$$\tau_t = -\tfrac{1}{2} A, \tau_u = 0, \tau_{xx} = 0, \tau_{tt} = 0, \Longrightarrow \tau = h\left(x, y\right) t + j\left(x, y\right) \Longrightarrow \tau = -\tfrac{1}{2} At + j\left(x, y\right),$$

$$\tau_x = j_x \Longrightarrow j_{xx} = 0 \Longrightarrow j\left(x, y\right) = c\left(y\right) x + d\left(y\right),$$

$$\Longrightarrow \tau = -\tfrac{1}{2} At + c\left(y\right) x + d\left(y\right). \tag{3.94}$$

Since $\xi_t = \tau_x$, we have

$$a\left(y\right) = c\left(y\right), \tag{3.95}$$

Plug in equation (e_{10}), we get

$$\cot x a\left(y\right) + \csc^2 x \left[a^{''}\left(y\right) x + d^{''}\left(y\right)\right] = 0, \tag{3.96}$$

This is possible only if

$$a\left(y\right) = 0, \tag{3.97}$$

and

$$d^{''}\left(y\right) = 0 \Longrightarrow d\left(y\right) = \alpha y + \beta, \tag{3.98}$$

So

$$\xi = -\tfrac{1}{2} Ax + b\left(y\right), \tag{3.99}$$

$$\tau = -\tfrac{1}{2} At + \alpha y + \beta. \tag{3.100}$$

Plug in equation (e_{12}), we get

$$-\tfrac{1}{2}A\cot x+\csc^2 x\left(-\tfrac{1}{2}Ax+b\left(y\right)\right)+\csc^2 x\left(b^{''}\left(y\right)\right)=0, \tag{3.101}$$

This is possible only if

$$A=0, \tag{3.102}$$

$$\Longrightarrow \tau_t=0, \tag{3.103}$$

which gives the minimal symmetry algebra.

Case2:

$f\left(u\right)$ is cubic, i.e. $f\left(u\right)=u^3+au^2+bu+c,$

$f_u=3u^2+2au+b,$

$f_{uu}=6u+2a,$

$f_{uuu}=6,$

$f_{uuuu}=0.$

From equation (3.75), we get

$$\phi_u=\left(\tfrac{2\cdot 6^3+0-0}{-2\cdot 6^3-0+0}\right)\tau_t\Longrightarrow\phi_u=-\tau_t. \tag{3.104}$$

From equation (3.76) we get

$$\phi=\tfrac{-(6u+2a)}{6}\tau_t\Longrightarrow\phi=-\left(u+\tfrac{a}{3}\right)\tau_t. \tag{3.105}$$

Also

$$\phi_{xu}=-\tau_{xt}=-\xi_{xx}, \tag{3.106}$$

$$\phi_{yu}=-\tau_{yt}=-\xi_{xy}, \tag{3.107}$$

$$\phi_{tu}=-\tau_{tt}=-\xi_{xt}. \tag{3.108}$$

Now

$$(3.22)\Longrightarrow\tau_{tt}+2\phi_{tu}=0\Longrightarrow\tau_{tt}-2\tau_{tt}=0\Longrightarrow\tau_{tt}=0\Longrightarrow\xi_{xt}=0, \tag{3.109}$$

$$(3.26)\Longrightarrow\tau_{yt}-2\phi_{yu}=0\Longrightarrow\tau_{yt}+2\tau_{yt}=0\Longrightarrow\tau_{yt}=0\Longrightarrow\xi_{xy}=0, \tag{3.110}$$

$$(3.28)\Longrightarrow\tau_{xt}+2\phi_{xu}=0\Longrightarrow\tau_{xt}-2\tau_{xt}=0\Longrightarrow\tau_{xt}=0\Longrightarrow\xi_{xx}=0. \tag{3.111}$$

Also

$\phi_{uu}=0$, implies that ϕ is linear in u i.e.

$$\phi=A\left(x,y,t\right)u+B\left(x,y,t\right), \tag{3.112}$$

$$\Longrightarrow\phi_u=A\left(x,y,t\right),$$

$$\Longrightarrow\tau_t=-A\left(x,y,t\right). \tag{3.113}$$

since $\tau_{tt}=0\Longrightarrow A=A\left(x,y\right),$

$\tau_{yt}=0\Longrightarrow A=A\left(x\right),$

$$\tau_{xt}=0\Longrightarrow A=c, \tag{3.114}$$

$$\Longrightarrow\phi=\left(u+\tfrac{a}{3}\right)A, \tag{3.115}$$

$$\Longrightarrow \phi_u = A. \tag{3.116}$$

Plug in equation (e_{13}),

$$\Longrightarrow f(A) - \left(u + \tfrac{a}{3}\right) A f_u - 0 + 0 + 2fA - 0 - 0 = 0 \Longrightarrow A\left(3f - \left(u + \tfrac{a}{3}\right) f_u\right) = 0,$$

$$\Longrightarrow A\left(3\left(u^3 + au^2 + bu + c\right) - \left(u + \tfrac{a}{3}\right)\left(3u^2 + 2au + b\right)\right) = 0,$$

$$\Longrightarrow A\left(3u^3 + 3au^2 + 3bu + 3c - 3u^3 - 2au^2 - bu - au^2 - \tfrac{2}{3}a^2u - \tfrac{ab}{3}\right) = 0,$$

$$\Longrightarrow A\left(\left(2b - \tfrac{2}{3}a^2\right)u + \left(3c - \tfrac{ab}{3}\right)\right) = 0,$$

$$\Longrightarrow \left(b - \tfrac{1}{3}a^2\right) = 0 \Longrightarrow b = \tfrac{1}{3}a^2 \text{ and } 3c - \tfrac{ab}{3} = 0 \Longrightarrow c = \tfrac{ab}{9},$$

$$\Longrightarrow c = \tfrac{a^3}{27}, \tag{3.117}$$

So: we must have

$$f(u) = u^3 + au^2 + \tfrac{a^2}{3}u + \tfrac{a^3}{27},$$

$$\Longrightarrow f(u) = \left(u + \tfrac{a}{3}\right)^3, \tag{3.118}$$

$$\Longrightarrow f(u) \text{ is a perfect cube}$$

So far we have

$$\xi_x = -A, \xi_u = 0, \xi_{xx} = 0, \xi_{tt} = 0 \Longrightarrow \xi = f(y,t)x + g(y,t) \Longrightarrow \xi = -Ax + g(y,t),$$

$$\xi_t = g_t \Longrightarrow g_{tt} = 0 \Longrightarrow g(y,t) = a(y)t + b(y),$$

$$\Longrightarrow \xi = -Ax + a(y)t + b(y). \tag{3.119}$$

Also

$$\tau_t = -A, \tau_u = 0, \tau_{xx} = 0, \tau_{tt} = 0, \Longrightarrow \tau = h(x,y)t + j(x,y) \Longrightarrow \tau = -At + j(x,y),$$

$$\tau_x = j_x \Longrightarrow j_{xx} = 0 \Longrightarrow j(x,y) = c(y)x + d(y),$$

$$\Longrightarrow \tau = -At + c(y)x + d(y). \tag{3.120}$$

Since $\xi_t = \tau_x$, we have

$$a(y) = c(y), \tag{3.121}$$

Plug in equation (e_{10}), we get

$$\cot x a(y) + \csc^2 x \left[a''(y)x + d''(y)\right] = 0, \tag{3.122}$$

This is possible only if

$$a(y) = 0, \tag{3.123}$$

and

$$d''(y) = 0 \Longrightarrow d(y) = \alpha y + \beta, \tag{3.124}$$

So

$$\xi = -Ax + b(y), \tag{3.125}$$

$$\tau = -At + \alpha y + \beta. \tag{3.126}$$

Plug in equation (e_{12}), we get

$$\cot x A + \csc^2 x(-Ax + b(y)) + \csc^2 x\left(b''(y)\right) = 0, \tag{3.127}$$

This is possible only if

$$A = 0, \tag{3.128}$$

$$\Longrightarrow \tau_t = 0, \tag{3.129}$$

which gives the minimal symmetry algebra.

Case3:

$$f(u) = ae^{bu} + cu + d, \quad \text{where } a, b \neq 0$$

$$f_u = abe^{bu} + c,$$

$$f_{uu} = ab^2 e^{bu},$$

$$f_{uuu} = ab^3 e^{bu},$$

$$f_{uuuu} = ab^4 e^{bu},$$

$$f_{uuuuu} = ab^5 e^{bu},$$

$$2f_{uuu}^3 + 2f_{uu}^2 f_{uuuuu} - 4f_{uu}f_{uuu}f_{uuuu} = 2a^3b^9e^{3bu} + 2a^2b^4e^{2bu}ab^5e^{bu} - 4a^3b^9e^{3bu} = 0.$$

From equation (3.75), we get

$$\phi_u = (0)\,\tau_t \Longrightarrow \phi_u = 0 \Longrightarrow \phi = B(x, y, t). \tag{3.130}$$

From equation (3.76), we get

$$\phi = \tfrac{-ab^2e^{bu}}{ab^3e^{bu}} 2\tau_t \Longrightarrow \phi = -\tfrac{2}{b}\tau_t \Longrightarrow \tau_t = -\tfrac{b}{2}\phi. \tag{3.131}$$

Now

$$(3.22) \Longrightarrow \tau_{tt} + 2\phi_{tu} = 0 \Longrightarrow \tau_{tt} = 0 \Longrightarrow \xi_{xt} = 0, \tag{3.132}$$

$$(3.26) \Longrightarrow \tau_{yt} - 2\phi_{yu} = 0 \Longrightarrow \tau_{yt} = 0 \Longrightarrow \xi_{xy} = 0, \tag{3.133}$$

$$(3.28) \Longrightarrow \tau_{xt} + 2\phi_{xu} = 0 \Longrightarrow \tau_{xt} = 0 \Longrightarrow \xi_{xx} = 0. \tag{3.134}$$

Also

$$\tau_{tt} = 0 \Longrightarrow \phi_t = 0 \Longrightarrow \phi = B(x, y),$$

$$\tau_{yt} = 0 \Longrightarrow \phi_y = 0 \Longrightarrow \phi = B(x),$$

$$\tau_{xt} = 0 \Longrightarrow \phi_x = 0 \Longrightarrow \phi = B, \quad \text{where } B \text{ is some constant} \tag{3.135}$$

So

$$\tau_t = -\tfrac{b}{2}B. \tag{3.136}$$

Plug in equation (e_{13}),

$$\Longrightarrow 0 - B\left(abe^{bu} + c\right) - 0 + 0 + 2\left(\tfrac{b}{2}B\right)\left(ae^{bu} + cu + d\right) - 0 - 0 = 0,$$

$$\Longrightarrow -abBe^{bu} - Bc + abBe^{bu} + cbBu + dbB = 0,$$

$$\Longrightarrow -Bc + cbBu + dbB = 0,$$

$$\Longrightarrow cbB = 0, \tag{3.137}$$

and

$$Bc = dbB, \tag{3.138}$$

If $B = 0$, then $\phi = 0 \Longrightarrow \tau_t = 0$, which gives the minimal symmetry algebra

And if $B \neq 0$, then

$$cb = 0, \tag{3.139}$$

and

$$c = db, \tag{3.140}$$

Since $b \neq 0$, then we must have

$c = 0$, which means that $d = 0$.

So: we must have

$$f(u) = ae^{bu}. \tag{3.141}$$

So far we have

$$\xi_x = -\tfrac{b}{2}B, \xi_u = 0, \xi_{xx} = 0, \xi_{tt} = 0 \Longrightarrow \xi = -\tfrac{b}{2}Bx + g(y,t),$$

$$\xi_t = g_t \Longrightarrow g_{tt} = 0 \Longrightarrow g(y,t) = a(y)t + b(y),$$

$$\Longrightarrow \xi = -\tfrac{b}{2}Bx + a(y)t + b(y). \tag{3.142}$$

Also

$$\tau_t = -\tfrac{b}{2}B, \tau_u = 0, \tau_{xx} = 0, \tau_{tt} = 0, \Longrightarrow \tau = -\tfrac{b}{2}Bt + j(x,y),$$

$$\tau_x = j_x \Longrightarrow j_{xx} = 0 \Longrightarrow j(x,y) = c(y)x + d(y),$$

$$\Longrightarrow \tau = -\tfrac{b}{2}Bt + c(y)x + d(y). \tag{3.143}$$

Since $\xi_t = \tau_x$, we have

$$a(y) = c(y). \tag{3.144}$$

Plug in equation (e_{10}), we get

$$\cot x a(y) + \csc^2 x \left[a''(y)x + d''(y)\right] = 0,$$

This is possible only if

$$a(y) = 0, \tag{3.145}$$

and

$$d''(y) = 0 \Longrightarrow d(y) = \alpha y + \beta. \tag{3.146}$$

So

$$\xi = -\tfrac{b}{2}Bx + b(y), \tag{3.147}$$

$$\tau = -\tfrac{b}{2}Bt + \alpha y + \beta. \tag{3.148}$$

Plug in equation (e_{12}), we get

$$\tfrac{b}{2}B\cot x + \csc^2 x\left(-\tfrac{b}{2}Bx + b(y)\right) + \csc^2 x\left(b''(y)\right) = 0,$$

This is possible only if

$$B = 0, \tag{3.149}$$

$\Longrightarrow \tau_t = 0,$

which gives the minimal symmetry algebra.

Case4:

$$f(u) = (u+b)^m + cu + d, \quad \text{where } m \neq 0,1,2,3,$$
$$f_u = m(u+b)^{m-1} + c,$$
$$f_{uu} = m(m-1)(u+b)^{m-2},$$
$$f_{uuu} = m(m-1)(m-2)(u+b)^{m-3},$$
$$f_{uuuu} = m(m-1)(m-2)(m-3)(u+b)^{m-4},$$
$$f_{uuuuu} = m(m-1)(m-2)(m-3)(m-4)(u+b)^{m-5}.$$

From equation (3.16), we get

$$\phi = \left(\frac{2f_{uu}f_{uuu}}{f_{uu}f_{uuuu} - 2f_{uuu}^2}\right)\tau_t,$$
$$\Longrightarrow \phi = \left(\frac{2m(m-1)(u+b)^{m-2}\cdot m(m-1)(m-2)(u+b)^{m-3}}{m(m-1)(u+b)^{m-2}\cdot m(m-1)(m-2)(m-3)(u+b)^{m-4} - 2\cdot m^2(m-1)^2(m-2)^2(u+b)^{2m-6}}\right)\tau_t,$$
$$\Longrightarrow \phi = \left(\frac{2(u+b)^{2m-5}}{(u+b)^{2m-6}[(m-3)-2(m-2)]}\right)\tau_t,$$
$$\Longrightarrow \phi = \left(\frac{-2}{m-1}\right)(u+b)\tau_t, \tag{3.150}$$
$$\Longrightarrow \phi_u = \left(\frac{-2}{m-1}\right)\tau_t. \tag{3.151}$$

Now

$$(3.22) \Longrightarrow \tau_{tt} + \left(\frac{-4}{m-1}\right)\tau_{tt} = 0 \Longrightarrow \left(\frac{m-5}{m-1}\right)\tau_{tt} = 0 \Longrightarrow \tau_{tt} = 0 \Longrightarrow \xi_{xt} = 0, \quad \text{for } m \neq 5,$$
$$(3.26) \Longrightarrow \tau_{yt} + \left(\frac{4}{m-1}\right)\tau_{yt} = 0 \Longrightarrow \left(\frac{m+3}{m-1}\right)\tau_{yt} = 0 \Longrightarrow \tau_{yt} = 0 \Longrightarrow \xi_{xy} = 0, \quad \text{for } m \neq -3,$$
$$(3.28) \Longrightarrow \tau_{xt} + \left(\frac{-4}{m-1}\right)\tau_{xt} = 0 \Longrightarrow \left(\frac{m-5}{m-1}\right)\tau_{xt} = 0 \Longrightarrow \tau_{xt} = 0 \Longrightarrow \xi_{xx} = 0, \quad \text{for } m \neq 5.$$

Also

$\phi_{uu} = 0$ implies that ϕ is linear in u i.e.

$$\phi = A(x,y,t)u + B(x,y,t),$$
$$\Longrightarrow \phi_u = A(x,y,t) \Longrightarrow \left(\frac{-2}{m-1}\right)\tau_t = A(x,y,t), \tag{3.152}$$

since $\tau_{tt} = 0 \Longrightarrow A = A(x,y)$,

$$\tau_{yt} = 0 \Longrightarrow A = A(x),$$
$$\tau_{xt} = 0 \Longrightarrow A = c, \tag{3.153}$$
$$\Longrightarrow \phi = \left(\frac{-2}{m-1}\right)(u+b)\cdot A\left(\frac{m-1}{-2}\right),$$
$$\Longrightarrow \phi = A(u+b), \tag{3.154}$$
$$\Longrightarrow \phi_u = A. \tag{3.155}$$

Plug in equation (e_{13}),

$$\Longrightarrow f(A) - (u+b)Af_u - 0 + 0 + 2\left(\frac{m-1}{2}\right)fA - 0 - 0 = 0 \Longrightarrow A(f - (u+b)f_u + (m-1)f) = 0,$$

$\Longrightarrow A\left(\left(u+b\right)^{m}+cu+d-m\left(u+b\right)^{m}-c\left(u+b\right)+\left(m-1\right)\left(u+b\right)^{m}+c\left(m-1\right)u+d\left(m-1\right)\right)=$
0,

$\Longrightarrow A\left(\left(u+b\right)^{m}+cu+d-m\left(u+b\right)^{m}-cu-cb+m\left(u+b\right)^{m}-\left(u+b\right)^{m}+cmu-cu+dm-d\right)=$
0,

$\Longrightarrow A\left(\left(cm-c\right)u+dm-cb\right)=0,$

$\Longrightarrow\left(cm-c\right)=0\Longrightarrow c\left(m-1\right)=0\Longrightarrow c=0,\quad \text{since } m\neq 1,$

and $dm=0\Longrightarrow d=0$, since $m\neq 0$,

So: we must have

$$f\left(u\right)=\left(u+b\right)^{m}. \tag{3.156}$$

So far we have

$$\xi_{x}=-\left(\tfrac{m-1}{2}\right)A,\xi_{u}=0,\xi_{xx}=0,\xi_{tt}=0\Longrightarrow\xi=-\left(\tfrac{m-1}{2}\right)Ax+g\left(y,t\right),$$

$$\xi_{t}=g_{t}\Longrightarrow g_{tt}=0\Longrightarrow g\left(y,t\right)=a\left(y\right)t+b\left(y\right),$$

$$\Longrightarrow\xi=-\left(\tfrac{m-1}{2}\right)Ax+a\left(y\right)t+b\left(y\right). \tag{3.157}$$

Also

$$\tau_{t}=-\left(\tfrac{m-1}{2}\right)A,\tau_{u}=0,\tau_{xx}=0,\tau_{tt}=0,\Longrightarrow\tau=-\left(\tfrac{m-1}{2}\right)At+j\left(x,y\right),$$

$$\tau_{x}=j_{x}\Longrightarrow j_{xx}=0\Longrightarrow j\left(x,y\right)=c\left(y\right)x+d\left(y\right),$$

$$\Longrightarrow\tau=-\left(\tfrac{m-1}{2}\right)At+c\left(y\right)x+d\left(y\right). \tag{3.158}$$

Since $\xi_{t}=\tau_{x}$, we have

$$a\left(y\right)=c\left(y\right). \tag{3.159}$$

Plug in equation $\left(e_{10}\right)$, we get

$$\cot xa\left(y\right)+\csc^{2}x\left[a^{''}\left(y\right)x+d^{''}\left(y\right)\right]=0,$$

This is possible only if

$$a\left(y\right)=0, \tag{3.160}$$

and

$$d^{''}\left(y\right)=0\Longrightarrow d\left(y\right)=\alpha y+\beta, \tag{3.161}$$

So

$$\xi=-\left(\tfrac{m-1}{2}\right)Ax+b\left(y\right), \tag{3.162}$$

$$\tau=-\left(\tfrac{m-1}{2}\right)At+\alpha y+\beta. \tag{3.163}$$

Plug in equation $\left(e_{12}\right)$, we get

$$\left(\tfrac{m-1}{2}\right)A\cot x+\csc^{2}x\left(-\left(\tfrac{m-1}{2}\right)Ax+b\left(y\right)\right)+\csc^{2}x\left(b^{''}\left(y\right)\right)=0,$$

This is possible only if

$$A=0\Longrightarrow\tau=\alpha y+\beta\Longrightarrow\tau_{t}=0,$$

which gives the minimal symmetry algebra.

For the case where $m = -3 \Longrightarrow f(u) = (u+b)^{-3} + cu + d,$ (3.164)

From equation (3.151), we get

$$\phi_u = \frac{1}{2}\tau_t. \tag{3.165}$$

From equation (3.150), we get

$$\phi = \frac{1}{2}(u+b)\,\tau_t. \tag{3.166}$$

Now

$$(3.22) \Longrightarrow \tau_{tt} = 0 \Longrightarrow \xi_{xt} = 0, \tag{3.167}$$

$$(3.28) \Longrightarrow \tau_{xt} = 0 \Longrightarrow \xi_{xx} = 0. \tag{3.168}$$

Also

$\phi_{uu} = 0$, implies that ϕ is linear in u i.e.

$$\phi = A(x,y,t)\,u + B(x,y,t)\,,$$

$$\Longrightarrow \phi_u = A(x,y,t) \Longrightarrow \frac{1}{2}\tau_t = A(x,y,t)\,,$$

$$\Longrightarrow \tau_t = 2A(x,y,t)\,. \tag{3.169}$$

since $\tau_{tt} = 0 \Longrightarrow A = A(x,y)\,,$

$$\tau_{xt} = 0 \Longrightarrow A = A(y)\,,$$

$$\Longrightarrow \tau_t = 2A(y)\,. \tag{3.170}$$

Now

$$(e_{10}) \Longrightarrow \cot x\tau_x + \csc^2 x\tau_{yy} + 2\phi_{tu} = 0 \Longrightarrow \cot x\tau_x + \csc^2 x\tau_{yy} = 0,$$

$$(e_{10})_t = 0 \Longrightarrow \cot x\tau_{xt} + \csc^2 x\tau_{yyt} = 0 \Longrightarrow \tau_{yyt} = 0. \tag{3.171}$$

From equation (3.170)

$$\Longrightarrow \tau_{yt} = 2A^{'}(y) \Longrightarrow \tau_{yyt} = 2A^{''}(y) = 0,$$

$$\Longrightarrow A(y) = k_1 y + k_2, \tag{3.172}$$

$$\Longrightarrow \phi = (k_1 y + k_2)\,u + B(x,y,t)\,,$$

$$\Longrightarrow \phi_u = k_1 y + k_2 \Longrightarrow \frac{1}{2}\tau_t = k_1 y + k_2,$$

$$\Longrightarrow \tau_t = 2k_1 y + 2k_2, \tag{3.173}$$

$$\Longrightarrow \tau_{yt} = 2k_1. \tag{3.174}$$

So, we have

$$\tau = (2k_1 y + 2k_2)\,t + g(x,y)\,. \tag{3.175}$$

Since $\xi_t = \tau_x$, we have

$$\xi_t = 2k_1 y + 2k_2,$$

$$\Longrightarrow \xi = (2k_1 y + 2k_2)\,x + j(y,t)\,. \tag{3.176}$$

And since $\xi_{tt} = 0 \Longrightarrow j_{tt} = 0 \Longrightarrow j = a(y)\,t + b(y)\,,$

$$\Longrightarrow \xi = (2k_1 y + 2k_2)\,x + a(y)\,t + b(y)\,. \tag{3.177}$$

And since $\tau_{xx}=0 \Longrightarrow g_{xx}=0 \Longrightarrow g=c\left(y\right)x+d\left(y\right),$

$$\Longrightarrow \tau=\left(2k_1y+2k_2\right)t+c\left(y\right)x+d\left(y\right). \qquad (3.178)$$

$\xi_t=\tau_x$ implies that $a\left(y\right)=c\left(y\right).$

Plug in equation $\left(e_{10}\right)$, we get

$$\cot xa\left(y\right)+\csc^2 x\left[a^{''}\left(y\right)x+d^{''}\left(y\right)\right]=0,$$

This is possible only if

$$a\left(y\right)=0, \qquad (3.179)$$

and

$$d^{''}\left(y\right)=0 \Longrightarrow d\left(y\right)=\alpha y+\beta, \qquad (3.180)$$

So

$$\xi=\left(2k_1y+2k_2\right)x+b\left(y\right), \qquad (3.181)$$

$$\tau=\left(2k_1y+2k_2\right)t+\alpha y+\beta. \qquad (3.182)$$

Plug in equation $\left(e_{12}\right)$, we get

$$-\cot x\left(2k_1y+2k_2\right)+\csc^2 x\left(\left(2k_1y+2k_2\right)x+b\left(y\right)\right)+\csc^2 x\left(b^{''}\left(y\right)\right)=0,$$

This is possible only if

$$k_1y+k_2=0, \qquad (3.183)$$

$$\Longrightarrow \phi=B\left(x,y,t\right) \Longrightarrow \phi_u=0,$$

$$\Longrightarrow \tau_t=0,$$

which gives the minimal symmetry algebra.

For the case where $m=5 \Longrightarrow f\left(u\right)=\left(u+b\right)^5,$

From equation (3.151), we get

$$\phi_u=-\tfrac{1}{2}\tau_t. \qquad (3.184)$$

From equation (3.150), we get

$$\phi=-\tfrac{1}{2}\left(u+b\right)\tau_t. \qquad (3.185)$$

Now,

equation (3.26) $\Longrightarrow \tau_{yt}=0 \Longrightarrow \xi_{xy}=0.$ (3.186)

Also

$\phi_{uu}=0$, implies that ϕ is linear in u i.e.

$$\phi=A\left(x,y,t\right)u+B\left(x,y,t\right), \qquad (3.187)$$

$$\Longrightarrow \phi_u=A\left(x,y,t\right) \Longrightarrow -\tfrac{1}{2}\tau_t=A\left(x,y,t\right), \qquad (3.188)$$

$$\Longrightarrow \tau_t=-2A\left(x,y,t\right). \qquad (3.189)$$

since $\tau_{ty}=0 \Longrightarrow A_y=0 \Longrightarrow A=A\left(x,t\right),$

$$\Longrightarrow \tau_t = -2A\left(x,t\right). \tag{3.190}$$

Now,

$(e_{10}) \Longrightarrow \cot x\tau_x + \csc^2 x\tau_{yy} + 2A_t = 0,$

$(e_{10})_t = 0 \Longrightarrow \cot x\tau_{xt} + 0 + 2A_{tt} = 0,$

$$\Longrightarrow -2\cot xA_x + 2A_{tt} = 0,$$

$$\Longrightarrow A_{tt} - \cot xA_x = 0. \tag{3.191}$$

Also,

$(e_{12}) \Longrightarrow -\cot x\xi_x + \csc^2 x\xi + \csc^2 x\xi_{yy} - 2\phi_{xu} = 0,$

$(e_{12})_y = 0 \Longrightarrow \csc^2 x\xi_y + \csc^2 x\xi_{yyy} = 0,$

$$\Longrightarrow \xi_{yyy} + \xi_y = 0, \tag{3.192}$$

where $\xi = \xi\left(x,y,t\right).$

Now, since $\xi_{xy} = 0 \Longrightarrow \xi_y = \psi\left(y,t\right),$ (3.193)

So, equation (3.192) becomes

$$\psi_{yy} + \psi = 0 \Longrightarrow \psi\left(y,t\right) = F_1(t)\sin y + F_2(t)\cos y,$$

$$\Longrightarrow \xi_y = F_1(t)\sin y + F_2(t)\cos y, \tag{3.194}$$

$$\Longrightarrow \xi\left(x,y,t\right) = -F_1(t)\cos y + F_2(t)\sin y + g(x,t), \tag{3.195}$$

$$\Longrightarrow \xi_x = g_x \Longrightarrow A\left(x,t\right) = -\frac{g_x}{2}, \tag{3.196}$$

$$\Longrightarrow \xi_{xx} = g_{xx}, \tag{3.197}$$

$$\Longrightarrow \xi_t = -F_1^{'}(t)\cos y + F_2^{'}(t)\sin y + g_t, \tag{3.198}$$

$$\Longrightarrow \xi_{tt} = -F_1^{''}(t)\cos y + F_2^{''}(t)\sin y + g_{tt}. \tag{3.199}$$

From equation (3.6), we have

$$\Longrightarrow \xi_{tt} - \xi_{xx} = 0,$$

$$\Longrightarrow -F_1^{''}(t)\cos y + F_2^{''}(t)\sin y + g_{tt} - g_{xx} = 0, \tag{3.200}$$

This is possible only if

$$F_1^{''}(t) = 0 \Longrightarrow F_1(t) = c_1t + c_2, \tag{3.201}$$

and

$$F_2^{''}(t) = 0 \Longrightarrow F_2(t) = c_3t + c_4, \tag{3.202}$$

and

$$g_{tt} - g_{xx} = 0 \Longrightarrow g_{xtt} - g_{xxx} = 0 \Longrightarrow -2A_{tt} + 2A_{xx} = 0,$$

$$\Longrightarrow A_{tt} - A_{xx} = 0. \tag{3.203}$$

So

$$\xi\left(x,y,t\right) = -(c_1t + c_2)\cos y + (c_3t + c_4)\sin y + g(x,t). \tag{3.204}$$

Putting in equation (e_{13}) gives

$$A\left(u+b\right)^{5}-5\left(u+b\right)^{4}\left(Au+B\right)-\left(A_{xx}u+B_{xx}\right)+\left(A_{tt}u+B_{tt}\right)$$
$$-2\left(u+b\right)^{5}\tau_{t}-\csc^{2}x\left(A_{yy}u+B_{yy}\right)-\cot x\left(A_{x}u+B_{x}\right)=0, \tag{3.205}$$

Since $\tau_t = -2A\left(x,t\right)$, equation (3.205) becomes

$$5A\left(u+b\right)^{5}-5Au\left(u+b\right)^{4}-5B\left(u+b\right)^{4}-\left(A_{xx}u+B_{xx}\right)+\left(A_{tt}u+B_{tt}\right)$$
$$-\csc^{2}x\left(A_{yy}u+B_{yy}\right)-\cot x\left(A_{x}u+B_{x}\right)=0,$$
$$5\left(u+b\right)^{4}\left[bA-B\right]+\left[A_{tt}-A_{xx}-\cot xA_{x}\right]u+\left[B_{tt}-B_{xx}-\csc^{2}xB_{yy}-\cot xB_{x}\right]=0. \tag{3.206}$$

Differentiating equation (3.206) with respect to u,

$$\Longrightarrow 20\left(u+b\right)^{3}\left[Ab-B\right]+\left[A_{tt}-A_{xx}-\cot xA_{x}\right]=0, \tag{3.207}$$

Differentiating equation (3.207) with respect to u,

$$\Longrightarrow 60\left(u+b\right)^{2}\left[bA(x,t)-B(x,y,t)\right]=0, \tag{3.208}$$

which implies that

$$bA(x,t)=B(x,y,t), \tag{3.209}$$
$$\Longrightarrow B_{y}=0\Longrightarrow B=B(x,t). \tag{3.210}$$

Now, equation (3.206) becomes

$$\Longrightarrow \left[A_{tt}-A_{xx}-\cot xA_{x}\right]u+\left[B_{tt}-B_{xx}-\cot xB_{x}\right]=0, \tag{3.211}$$

Differentiating equation (3.211) with respect to u,

$$\Longrightarrow A_{tt}-A_{xx}-\cot xA_{x}=0, \tag{3.212}$$

which implies that

$$B_{tt}-B_{xx}-\cot xB_{x}=0, \tag{3.213}$$

Now, since $A_{tt}-A_{xx}=0$, equation (3.212) becomes

$$\cot xA_{x}=0\Longrightarrow A_{x}=0,$$
$$\Longrightarrow A=G(t). \tag{3.214}$$

Since $A_x=0\Longrightarrow A_{xx}=0\Longrightarrow A_{tt}=0\Longrightarrow G''(t)=0$,

$$\Longrightarrow G(t)=\alpha t+\beta,$$
$$\Longrightarrow A(t)=\alpha t+\beta. \tag{3.215}$$

So, we have

$$\Longrightarrow \phi=\left(\alpha t+\beta\right)u+b\left(\alpha t+\beta\right), \tag{3.216}$$
$$\Longrightarrow \phi_{u}=\alpha t+\beta\Longrightarrow -\tfrac{1}{2}\tau_{t}=\alpha t+\beta,$$
$$\Longrightarrow \tau_{t}=-2(\alpha t+\beta), \tag{3.217}$$
$$\Longrightarrow \xi_{x}=-2(\alpha t+\beta). \tag{3.218}$$

From equation (3.204), we have

$$\xi\left(x,y,t\right)=-(c_{1}t+c_{2})\cos y+(c_{3}t+c_{4})\sin y+g(x,t), \tag{3.219}$$
$$\Longrightarrow \xi_{x}=g_{x}\Longrightarrow -\tfrac{g_{x}}{2}=A=\alpha t+\beta,$$

$$\Longrightarrow g_x = -2(\alpha t + \beta) \Longrightarrow g = -2(\alpha t + \beta)x + h(t),$$

$$\Longrightarrow \xi\,(x, y, t) = -(c_1 t + c_2)\cos y + (c_3 t + c_4)\sin y - 2(\alpha t + \beta)x + h(t). \tag{3.220}$$

From $\xi_{tt} - \xi_{xx} = 0$, equation (3.220) implies that

$$h^{''}(t) = 0 \Longrightarrow h(t) = h_1 t + h_2, \tag{3.221}$$

$$\Longrightarrow \xi = -(c_1 t + c_2)\cos y + (c_3 t + c_4)\sin y - 2(\alpha t + \beta)x + h_1 t + h_2, \tag{3.222}$$

$$\Longrightarrow \xi_t = -c_1 \cos y + c_3 \sin y - 2\alpha x + h_1, \tag{3.223}$$

$$\Longrightarrow \tau_x = -c_1 \cos y + c_3 \sin y - 2\alpha x + h_1. \tag{3.224}$$

Since $\tau_t = -2(\alpha t + \beta)$, we have

$$\tau = -2(\alpha \tfrac{t^2}{2} + \beta t) + K(x, y), \tag{3.225}$$

$$\Longrightarrow \tau_x = K_x = -c_1 \cos y + c_3 \sin y - 2\alpha x + h_1,$$

$$\Longrightarrow K(x, y) = -c_1 x \cos y + c_3 x \sin y - \alpha x^2 + h_1 x + p(y),$$

$$\Longrightarrow \tau = -2(\alpha \tfrac{t^2}{2} + \beta t) - c_1 x \cos y + c_3 x \sin y - \alpha x^2 + h_1 x + p(y). \tag{3.226}$$

Putting in equation (e_{10}), gives

$$\cot x[-c_1 \cos y + c_3 \sin y - 2\alpha x + h_1] + \csc^2 x[c_1 x \cos y - c_3 x \sin y + p^{''}(y)] + 2\alpha = 0, \tag{3.227}$$

Copmaring the coefficients, we get

$$c_1 \cos y - c_3 \sin y + h_1 = 0, \tag{3.228}$$

which means that

$$c_1 = 0,\ c_3 = 0 \text{ and } h_1 = 0, \tag{3.229}$$

and

$$2\alpha = 0 \Longrightarrow \alpha = 0, \tag{3.230}$$

and

$$p^{''}(y) = 0 \Longrightarrow p(y) = k_1 y + k_2. \tag{3.231}$$

So, we have

$$\Longrightarrow A = \beta, \tag{3.232}$$

$$\Longrightarrow \phi = \beta(u + b), \tag{3.233}$$

$$\Longrightarrow \phi_u = \beta \Longrightarrow -\tfrac{1}{2}\tau_t = \beta, \tag{3.234}$$

$$\Longrightarrow \tau = -2\beta t + k_1 y + k_2, \tag{3.235}$$

$$\Longrightarrow \xi = -c_2 \cos y + c_4 \sin y - 2\beta x + h_2. \tag{3.236}$$

Next, we note that

$$\theta_u = 0 \Longrightarrow \theta = \theta\,(x, y, t)\,, \tag{3.237}$$

$$(e_7)_t = 0 \Longrightarrow \csc^2 x \tau_{yt} - \theta_{tt} = 0,$$

$$\Longrightarrow \theta_{tt} = 0, \tag{3.238}$$

which implies that θ is linear in t,

$$\Longrightarrow \theta = E(x,y)\,t + H(x,y)\,, \tag{3.239}$$

$$(e_6)_t = 0 \Longrightarrow \cot x \xi_t - \tau_{tt} + \theta_{ty} = 0 \Longrightarrow \theta_{ty} = 0 \Longrightarrow E_y = 0 \Longrightarrow E = E(x)\,,$$

$$\Longrightarrow \theta = E(x)\,t + H(x,y)\,, \tag{3.240}$$

$$(e_3)_t = 0 \Longrightarrow \csc^2 x \xi_{yt} + \theta_{xt} = 0 \Longrightarrow \theta_{xt} = 0 \Longrightarrow E_x = 0 \Longrightarrow E = c_5. \tag{3.241}$$

To determine c_5, we note that equation (e_7), becomes

$$k_1 \csc^2 x - c_5 = 0 \Longrightarrow k_1 = c_5 = 0. \tag{3.242}$$

To determine $H(x,y)$, we note that

$$(e_3) \Longrightarrow H_x = -\csc^2 x\,[c_2 \sin y + c_4 \cos y]\,, \tag{3.243}$$

Integrating equation (3.243), we get

$$\Longrightarrow H = \cot x\,[c_2 \sin y + c_4 \cos y] + q(y)\,, \tag{3.244}$$

$$\Longrightarrow H_y = \cot x\,[c_2 \cos y - c_4 \sin y] + q^{'}(y)\,. \tag{3.245}$$

So, (e_6) becomes

$$-\cot x\,[c_2 \cos y - c_4 \sin y + 2\beta x - h_2] + 2\beta = -\cot x\,[c_2 \cos y - c_4 \sin y] - q^{'}(y)\,,$$

$$\Longrightarrow q^{'}(y) = \cot x\,[2\beta x - h_2] - 2\beta, \tag{3.246}$$

$$\Longrightarrow q(y) = y \cot x\,[2\beta x - h_2] - 2\beta y + k_4,$$

$$\Longrightarrow \theta = \cot x\,[c_2 \sin y + c_4 \cos y] + y \cot x\,[2\beta x - h_2] - 2\beta y + k_4. \tag{3.247}$$

So, (e_3) becomes

$$\Longrightarrow \csc^2 x[c_2 \sin y + c_4 \cos y] - \csc^2 x[c_2 \sin y + c_4 \cos y] - y \csc^2 x\,[2\beta x - h_2]$$
$$+2\beta y \cot x = 0,$$

$$\Longrightarrow -2\beta x y \csc^2 x + h_2 y \csc^2 x + 2\beta y \cot x = 0, \tag{3.248}$$

which implies that

$$\beta = 0 \text{ and } h_2 = 0, \tag{3.249}$$

which means that

$$A = 0 \Longrightarrow \tau_t = 0, \tag{3.250}$$

which gives the minimal symmetry algebra.

Case5:

$$f(u) = \ln(u+b) + cu + d,$$

$$f_u = (u+b)^{-1} + c,$$

$$f_{uu} = -(u+b)^{-2}\,,$$

$$f_{uuu} = 2(u+b)^{-3}\,,$$

$$f_{uuuu} = -6(u+b)^{-4}\,,$$

$$f_{uuuuu} = 24(u+b)^{-5}\,.$$

From equation (3.75), we get

$$\phi_u = \left(\frac{2\cdot 8(u+b)^{-9}+2\cdot(u+b)^{-4}\cdot 24(u+b)^{-5}-4\cdot 12(u+b)^{-9}}{-(u+b)^{-4}\cdot 24(u+b)^{-5}+4\cdot 12(u+b)^{-9}-2\cdot 8(u+b)^{-9}}\right)\tau_t,$$

$$\Longrightarrow \phi_u = \left(\frac{16+48-48}{-24+48-16}\right)\tau_t,$$

$$\Longrightarrow \phi_u = 2\tau_t. \tag{3.251}$$

From equation (3.76) we get

$$\phi = \frac{(u+b)^{-2}}{2(u+b)^{-3}}\left(4\tau_t\right).$$

$$\Longrightarrow \phi = 2\left(u+b\right)\tau_t. \tag{3.252}$$

Now

$$(3.22) \Longrightarrow \tau_{tt}+4\tau_{tt} = 0 \Longrightarrow \tau_{tt} = 0 \Longrightarrow \xi_{xt} = 0, \tag{3.253}$$

$$(3.26) \Longrightarrow \tau_{yt}-4\tau_{yt} = 0 \Longrightarrow \tau_{yt} = 0 \Longrightarrow \xi_{xy} = 0, \tag{3.254}$$

$$(3.28) \Longrightarrow \tau_{xt}+4\tau_{xt} = 0 \Longrightarrow \tau_{xt} = 0 \Longrightarrow \xi_{xx} = 0. \tag{3.255}$$

Also

$\phi_{uu} = 0$, implies that ϕ is linear in u, i.e.

$$\phi = A\left(x,y,t\right)u + B\left(x,y,t\right),$$

$$\Longrightarrow \phi_u = A\left(x,y,t\right),$$

$$\Longrightarrow \tau_t = \frac{1}{2}A\left(x,y,t\right). \tag{3.256}$$

since $\tau_{tt} = 0 \Longrightarrow A = A\left(x,y\right),$

$$\tau_{yt} = 0 \Longrightarrow A = A\left(x\right),$$

$$\tau_{xt} = 0 \Longrightarrow A = c,$$

$$\Longrightarrow \phi = \left(u+b\right)A, \tag{3.257}$$

$$\Longrightarrow \phi_u = A. \tag{3.258}$$

Plug in equation (e_{13}),

$$\Longrightarrow f\left(A\right) - \left(u+b\right)Af_u - 0 + 0 - 2\tfrac{1}{2}fA - 0 - 0 = 0 \Longrightarrow Af_u\left(u+b\right) = 0,$$

$$\Longrightarrow A\left(u+b\right)\left(\frac{1}{(u+b)}+c\right) = 0 \Longrightarrow A + cA\left(u+b\right) = 0,$$

$$\Longrightarrow A\left(1+c\left(u+b\right)\right) = 0 \Longrightarrow A\left(1+cu+cb\right) = 0,$$

$$\Longrightarrow c = 0 \Longrightarrow A = 0 \Longrightarrow \tau_t = 0,$$

which gives the minimal symmetry algebra.

Case6:

$$f\left(u\right) = \left[\left(u+b\right)\ln\left(u+b\right) - \left(u+b\right)\right] + cu + d,$$

$$f_u = \ln\left(u+b\right) + c,$$

$$f_{uu} = \left(u+b\right)^{-1},$$

$$f_{uuu} = -\left(u+b\right)^{-2},$$

$$f_{uuuu} = 2\left(u+b\right)^{-3},$$

$$f_{uuuuu} = -6\left(u+b\right)^{-4}.$$

From equation (3.16), we get

$$f_{uu}f_{uuuu} - 2f_{uuu}^2 = 2\left(u+b\right)^{-1}\left(u+b\right)^{-3} - 2\left(u+b\right)^{-4} = 0,$$

$$\Longrightarrow \left(f_{uu}f_{uuu}\right)\tau_t = 0 \Longrightarrow \tfrac{1}{(u+b)^3}\tau_t = 0 \Longrightarrow \tau_t = 0,$$

which gives the minimal symmetry algebra.

3.2 Symmetry reductions and invariant solutions

One of the main purposes for calculating symmetries of a differential equation is to use them for obtaining symmetry reductions and finding exact solutions. In this section, we will use the symmetries calculated in the previous sections to obtain a few exact solutions of equation (3.1). More exact solutions can be calculated in a similar way.

Our main goal is to derive exact solutions of equation (3.1) by reducing it to ordinary differential equations. This can be achieved with the use of Lie point symmetries admitted by equation (3.1). It is well known that the reduction of a partial differential equation with respect to n-dimensional solvable subalgebra of its Lie symmetry algebra leads to reducing the number of independent variables by n.

3.2.1 Reduction of The General Case

We are going to reduce the following general equation

$$u_{tt} = u_{xx} + (\cot x)\, u_x + \left(\csc^2 x\right) u_{yy} + f\,(u)\,. \tag{3.259}$$

3.2.1.1 Reduction Using Subalgebra $\langle X_3 + X_4, X_4\rangle = \left\langle \frac{\partial}{\partial t} + \frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right\rangle$

Starting with the first symmetry which can be written as

$$X = 0\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial t} + 0\frac{\partial}{\partial u}, \tag{3.260}$$

The characteristic system

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dt}{1} = \frac{du}{0},$$

From $\frac{dx}{0} = \frac{dy}{1}$, we have

$$x = c,$$

and from $\frac{dy}{1} = \frac{dt}{1}$, we get

$$y = t + c \Longrightarrow y - t = c,$$

and from $\frac{dt}{1} = \frac{du}{0}$, we get

$$u = c.$$

So, we get three invariants of $X\,(I) = 0$, and they are given by

$$q\,(x, y, t) = x, \tag{3.261}$$

$$v\,(x, y, t) = y - t, \tag{3.262}$$

$$P\,(q, v) = u. \tag{3.263}$$

Now:

$$u_t = P_q q_t + P_v v_t = -P_v, \tag{3.264}$$

$$u_{tt} = P_{vv}, \tag{3.265}$$

$$u_x = P_q q_x + P_v v_x = P_q, \tag{3.266}$$

$$u_{xx} = P_{qq}, \tag{3.267}$$

$$u_y = P_q q_y + P_v v_y = P_v, \tag{3.268}$$

$$u_{yy} = P_{vv}, \tag{3.269}$$

Substituting in equation (3.259), we get

$$P_{vv} = P_{qq} + (\cot q)\, P_q + \left(\csc^2 q\right) P_{vv} + f\,(P)\,. \tag{3.270}$$

The second symmetry X_4 is inherited by PDE (3.270), as it commutes with $X_3 + X_4$.

So, using the second symmetry $\overline{X} = X_4 = \frac{\partial}{\partial y}$,

$$X = \overline{X}\,(q)\,\tfrac{\partial}{\partial q} + \overline{X}\,(v)\,\tfrac{\partial}{\partial v} + \overline{X}\,(P)\,\tfrac{\partial}{\partial P},$$

$$\overline{X}\,(q) = \overline{X}\,(x) = 0,$$

$$\overline{X}\,(v) = \overline{X}\,(y-t) = 1,$$

$$\overline{X}\,(P) = \overline{X}\,(u) = 0,$$

So, the second symmetry can be written as

$$X = 0\tfrac{\partial}{\partial q} + \tfrac{\partial}{\partial v} + 0\tfrac{\partial}{\partial P}, \tag{3.271}$$

The characteristic system

$$\tfrac{dq}{0} = \tfrac{dv}{1} = \tfrac{dP}{0},$$

From $\frac{dq}{0} = \frac{dv}{1}$, we have

$$q = c,$$

and from $\frac{dv}{1} = \frac{dP}{0}$, we get

$$P = c,$$

So, the invariants are

$$\xi\,(q, v) = q, \tag{3.272}$$

$$\omega\,(\xi) = P, \tag{3.273}$$

Now:

$$P_v = \omega_\xi \xi_v = 0, \tag{3.274}$$

$$P_{vv} = 0, \tag{3.275}$$

$$P_q = \omega_\xi \xi_q = \omega^{'}, \tag{3.276}$$

$$P_{qq} = \omega^{''}, \tag{3.277}$$

Substituting in equation (3.270), we get the ODE

$$\omega^{''} + (\cot \xi)\, \omega^{'} + f\,(\omega) = 0. \tag{3.278}$$

3.2.1.2 Reduction Using Subalgebra $\langle X_3, X_4 \rangle = \left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial y} \right\rangle$

Starting with the first symmetry which can be written as

$$X = 0\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} + \frac{\partial}{\partial t} + 0\frac{\partial}{\partial u}, \tag{3.279}$$

The characteristic system

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{0},$$

From $\frac{dx}{0} = \frac{dt}{1}$, we have

$$x = c,$$

and from $\frac{dy}{0} = \frac{dt}{1}$, we get

$$y = c,$$

and from $\frac{dt}{1} = \frac{du}{0}$, we get

$$u = c,$$

So, we get three invariants

$$q\left(x, y, t\right) = x, \tag{3.280}$$

$$v\left(x, y, t\right) = y, \tag{3.281}$$

$$P\left(q, v\right) = u, \tag{3.282}$$

Now:

$$u_t = P_q q_t + P_v v_t = 0, \tag{3.283}$$

$$u_{tt} = 0, \tag{3.284}$$

$$u_x = P_q q_x + P_v v_x = P_q, \tag{3.285}$$

$$u_{xx} = P_{qq}, \tag{3.286}$$

$$u_y = P_q q_y + P_v v_y = P_v, \tag{3.287}$$

$$u_{yy} = P_{vv}, \tag{3.288}$$

Substituting in equation (3.259), we get

$$P_{qq} + \left(\cot q\right) P_q + \left(\csc^2 q\right) P_{vv} + f\left(P\right) = 0. \tag{3.289}$$

The second symmetry X_4 is inherited by PDE (3.289), as it commutes with X_3.

So, using the second symmetry $\overline{X} = X_4 = \frac{\partial}{\partial y}$,

$$X = \overline{X}\left(q\right) \frac{\partial}{\partial q} + \overline{X}\left(v\right) \frac{\partial}{\partial v} + \overline{X}\left(P\right) \frac{\partial}{\partial P},$$

$$\overline{X}\left(q\right) = \overline{X}\left(x\right) = 0,$$

$$\overline{X}\left(v\right) = \overline{X}\left(y\right) = 1,$$

$$\overline{X}\left(P\right) = \overline{X}\left(u\right) = 0,$$

So, the second symmetry can be written as

$$X = 0\frac{\partial}{\partial q} + \frac{\partial}{\partial v} + 0\frac{\partial}{\partial P}, \tag{3.290}$$

The characteristic system

$$\frac{dq}{0} = \frac{dv}{1} = \frac{dP}{0},$$

From $\frac{dq}{0} = \frac{dv}{1}$, we have

$$q = c,$$

and from $\frac{dv}{1} = \frac{dP}{0}$, we get

$$P = c,$$

So, the invariants are

$$\xi\,(q, v) = q, \tag{3.291}$$

$$\omega\,(\xi) = P, \tag{3.292}$$

Now:

$$P_v = \omega_\xi \xi_v = 0, \tag{3.293}$$

$$P_{vv} = 0, \tag{3.294}$$

$$P_q = \omega_\xi \xi_q = \omega^{'}, \tag{3.295}$$

$$P_{qq} = \omega^{''}, \tag{3.296}$$

Substituting in equation (3.289), we get the ODE

$$\omega^{''} + (\cot \xi)\,\omega^{'} + f\,(\omega) = 0. \tag{3.297}$$

3.2.1.3 Reduction Using Subalgebra $\langle X_3, X_1 \rangle = \left\langle \frac{\partial}{\partial t}, \cos y \frac{\partial}{\partial x} - \sin y \cot x \frac{\partial}{\partial y} \right\rangle$

Starting with the first symmetry which can be written as

$$X = 0\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} + \frac{\partial}{\partial t} + 0\frac{\partial}{\partial u}, \tag{3.298}$$

The characteristic system

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{0},$$

From $\frac{dx}{0} = \frac{dt}{1}$, we have

$$x = c,$$

and from $\frac{dy}{0} = \frac{dt}{1}$, we get

$$y = c,$$

and from $\frac{dt}{1} = \frac{du}{0}$, we get

$$u = c,$$

So, we get three invariants

$$q\,(x, y, t) = x, \tag{3.299}$$

$$v\,(x, y, t) = y, \tag{3.300}$$

$$P\,(q, v) = u, \tag{3.301}$$

Now:

$$u_t = P_q q_t + P_v v_t = 0, \tag{3.302}$$

$$u_{tt} = 0, \tag{3.303}$$

$$u_x = P_q q_x + P_v v_x = P_q, \tag{3.304}$$

$$u_{xx} = P_{qq}, \tag{3.305}$$

$$u_y = P_q q_y + P_v v_y = P_v, \tag{3.306}$$

$$u_{yy} = P_{vv}, \tag{3.307}$$

So, equation (3.259) becomes

$$P_{vv} = P_{qq} + (\cot q) P_q + \left(\csc^2 q\right) P_{vv} + f(P). \tag{3.308}$$

The second symmetry X_1 is inherited by PDE (3.308), as it commutes with X_3.

So, using the second symmetry $\overline{X} = X_1 = \cos y \frac{\partial}{\partial x} - \sin y \cot x \frac{\partial}{\partial y}$,

$$X = \overline{X}(q)\frac{\partial}{\partial q} + \overline{X}(v)\frac{\partial}{\partial v} + \overline{X}(P)\frac{\partial}{\partial P},$$

$$\overline{X}(q) = \overline{X}(x) = \cos v,$$

$$\overline{X}(v) = \overline{X}(y) = -\sin v \cot q,$$

$$\overline{X}(P) = \overline{X}(u) = 0,$$

So, the second symmetry can be written as

$$X = \cos v \frac{\partial}{\partial q} - \sin v \cot q \frac{\partial}{\partial v} + 0 \frac{\partial}{\partial P}, \tag{3.309}$$

The characteristic system

$$\frac{dq}{\cos v} = \frac{dv}{-\sin v \cot q} = \frac{dP}{0},$$

From $\frac{dq}{\cos v} = \frac{dv}{-\sin v \cot q}$, we have

$$-\sin v \cot q dq = \cos v dv \Longrightarrow \cot q dq = -\cot v dv,$$

$$\Longrightarrow \ln(\sin q) = -\ln(\sin v) + c \Longrightarrow \ln(\sin q \sin v) = c \Longrightarrow \sin q \sin v = c,$$

and from $\frac{dq}{\cos v} = \frac{dP}{0}$, we get

$$P = c,$$

So, the invariants are

$$\xi(q, v) = \sin q \sin v, \tag{3.310}$$

$$\omega(\xi) = P, \tag{3.311}$$

Now:

$$P_q = \omega_\xi \xi_q = \cos q \sin v \omega', \tag{3.312}$$

$$P_{qq} = \cos^2 q \sin^2 v \omega'' - \sin q \sin v \omega', \tag{3.313}$$

$$P_v = \omega_\xi \xi_v = \sin q \cos v \omega', \tag{3.314}$$

$$P_{vv} = \sin^2 q \cos^2 v \omega'' - \sin q \sin v \omega', \tag{3.315}$$

Substituting in equation (3.308), we get

$$\left(\cos^2 q \sin^2 v \omega'' - \sin q \sin v \omega'\right) + \cot q \left(\cos q \sin v \omega'\right)$$

$$+\csc^2 q\left(\sin^2 q\cos^2 v\omega^{''} - \sin q\sin v\omega^{'}\right) + f(\omega) = 0,$$

$$\Longrightarrow \left(\cos^2 q\sin^2 v + \csc^2 q\sin^2 q\cos^2 v\right)\omega^{''} - \left(\sin q\sin v - \cot q\cos q\sin v + \csc^2 q\sin q\sin v\right)\omega^{'}$$

$$+f(\omega) = 0,$$

$$\Longrightarrow \left(\left(1-\sin^2 q\right)\sin^2 v + \cos^2 v\right)\omega^{''} - \left(\xi - \xi\tfrac{\cos^2 q}{\sin^2 q} + \xi\tfrac{1}{\sin^2 q}\right)\omega^{'} + f(\omega) = 0,$$

$$\Longrightarrow \left(\sin^2 v - \sin^2 q\sin^2 v + \cos^2 v\right)\omega^{''} - \left(\xi - \xi\tfrac{\left(\cos^2 q - 1\right)}{\sin^2 q}\right)\omega^{'} + f(\omega) = 0,$$

$$\Longrightarrow \left(1 - \sin^2 q\sin^2 v\right)\omega^{''} - (\xi + \xi)\,\omega^{'} + f(\omega) = 0,$$

$$\Longrightarrow \left(1 - \xi^2\right)\omega^{''} - 2\xi\omega^{'} + f(\omega) = 0. \tag{3.316}$$

3.2.1.4 Reduction Using Subalgebra $\langle X_3, X_2\rangle = \left\langle \frac{\partial}{\partial t}, \sin y\frac{\partial}{\partial x} + \cos y\cot x\frac{\partial}{\partial y}\right\rangle$

Starting with the first symmetry which can be written as

$$X = 0\tfrac{\partial}{\partial x} + 0\tfrac{\partial}{\partial y} + \tfrac{\partial}{\partial t} + 0\tfrac{\partial}{\partial u}, \tag{3.317}$$

The characteristic system

$$\tfrac{dx}{0} = \tfrac{dy}{0} = \tfrac{dt}{1} = \tfrac{du}{0},$$

From $\frac{dx}{0} = \frac{dt}{1}$, we have

$$x = c,$$

and from $\frac{dy}{0} = \frac{dt}{1}$, we get

$$y = c,$$

and from $\frac{dt}{1} = \frac{du}{0}$, we get

$$u = c,$$

So, we get three invariants

$$q(x,y,t) = x, \tag{3.318}$$

$$v(x,y,t) = y, \tag{3.319}$$

$$P(q,v) = u, \tag{3.320}$$

Now:

$$u_t = P_q q_t + P_v v_t = 0, \tag{3.321}$$

$$u_{tt} = 0, \tag{3.322}$$

$$u_x = P_q q_x + P_v v_x = P_q, \tag{3.323}$$

$$u_{xx} = P_{qq}, \tag{3.324}$$

$$u_y = P_q q_y + P_v v_y = P_v, \tag{3.325}$$

$$u_{yy} = P_{vv}, \tag{3.326}$$

So, equation (3.259) becomes

$$P_{vv} = P_{qq} + (\cot q)\,P_q + \left(\csc^2 q\right)P_{vv} + f(P). \tag{3.327}$$

The second symmetry X_2 is inherited by PDE (3.327), as it commutes with X_3.

So, using the second symmetry $\overline{X} = X_2 = \sin y \frac{\partial}{\partial x} + \cos y \cot x \frac{\partial}{\partial y}$,

$$X = \overline{X}(q) \frac{\partial}{\partial q} + \overline{X}(v) \frac{\partial}{\partial v} + \overline{X}(P) \frac{\partial}{\partial P},$$

$$\overline{X}(q) = \overline{X}(x) = \sin v,$$

$$\overline{X}(v) = \overline{X}(y) = \cos v \cot q,$$

$$\overline{X}(P) = \overline{X}(u) = 0,$$

So, the second symmetry can be written as

$$X = \sin v \frac{\partial}{\partial q} + \cos v \cot q \frac{\partial}{\partial v} + 0 \frac{\partial}{\partial P}, \tag{3.328}$$

The characteristic system

$$\frac{dq}{\sin v} = \frac{dv}{\cos v \cot q} = \frac{dP}{0},$$

From $\frac{dq}{\sin v} = \frac{dv}{\cos v \cot q}$, we have

$$\cos v \cot q dq = \sin v dv \Longrightarrow \cot q dq = \tan v dv,$$

$$\Longrightarrow \ln(\sin q) = -\ln(\cos v) + c \Longrightarrow \ln(\sin q \cos v) = c \Longrightarrow \sin q \cos v = c,$$

and from $\frac{dq}{\sin v} = \frac{dP}{0}$, we get

$$P = c,$$

So, the invariants are

$$\xi(q, v) = \sin q \cos v, \tag{3.329}$$

$$\omega(\xi) = P, \tag{3.330}$$

Now:

$$P_q = \omega_\xi \xi_q = \cos q \cos v \omega^{'}, \tag{3.331}$$

$$P_{qq} = \cos^2 q \cos^2 v \omega^{''} - \sin q \cos v \omega^{'}, \tag{3.332}$$

$$P_v = \omega_\xi \xi_v = -\sin q \sin v \omega^{'}, \tag{3.333}$$

$$P_{vv} = \sin^2 q \sin^2 v \omega^{''} - \sin q \cos v \omega^{'}, \tag{3.334}$$

Substituting in equation (3.327), we get

$$\left(\cos^2 q \cos^2 v \omega^{''} - \sin q \cos v \omega^{'}\right) + \cot q \left(\cos q \cos v \omega^{'}\right)$$
$$+ \csc^2 q \left(\sin^2 q \sin^2 v \omega^{''} - \sin q \cos v \omega^{'}\right) + f(\omega) = 0,$$

$$\Longrightarrow \left(\cos^2 q \cos^2 v + \csc^2 q \sin^2 q \sin^2 v\right) \omega^{''} - \left(\sin q \cos v - \cot q \cos q \cos v + \csc^2 q \sin q \cos v\right) \omega^{'}$$
$$+ f(\omega) = 0,$$

$$\Longrightarrow \left(\left(1 - \sin^2 q\right) \cos^2 v + \sin^2 v\right) \omega^{''} - \left(\xi - \xi \frac{\cos^2 q}{\sin^2 q} + \xi \frac{1}{\sin^2 q}\right) \omega^{'} + f(\omega) = 0,$$

$$\Longrightarrow \left(\cos^2 v - \sin^2 q \cos^2 v + \sin^2 v\right) \omega^{''} - \left(\xi - \xi \frac{\left(\cos^2 q - 1\right)}{\sin^2 q}\right) \omega^{'} + f(\omega) = 0,$$

$$\Longrightarrow \left(1 - \sin^2 q \cos^2 v\right) \omega^{''} - (\xi + \xi) \omega^{'} + f(\omega) = 0,$$

$$\Longrightarrow \left(1 - \xi^2\right) \omega^{''} - 2\xi \omega^{'} + f(\omega) = 0. \tag{3.335}$$

3.2.2 Solutions of the equation when $f(u) = -e^u - 2$

We are going to reduce the following equation

$$u_{tt} = u_{xx} + (\cot x)\, u_x + \left(\csc^2 x\right) u_{yy} - e^u - 2. \tag{3.336}$$

3.2.2.1 Reduction Using $\langle X_3, X_4 \rangle$

Using section 3.2.1.2, equation (3.336) reduces to

$$\omega^{''} + (\cot \xi)\, \omega^{'} - e^{\omega} - 2 = 0, \tag{3.337}$$

Equation (3.337) has the symmetry

$$X = \sin \xi \frac{\partial}{\partial \xi} - 2 \cos \xi \frac{\partial}{\partial \omega}, \tag{3.338}$$

To find the first prolongation of X,

$$\eta^{[1]} = \eta_\xi + \left(\eta_\omega - \psi_\xi\right) \omega^{'} - \psi_\omega \omega^{'^2},$$

$$\eta^{[1]} = 2 \sin \xi - \cos \xi \omega^{'}, \tag{3.339}$$

So

$$X^{[1]} = \sin \xi \frac{\partial}{\partial \xi} - 2 \cos \xi \frac{\partial}{\partial \omega} + \left(2 \sin \xi - \cos \xi \omega^{'}\right) \frac{\partial}{\partial \omega^{'}}, \tag{3.340}$$

The characteristic system

$$\frac{d\xi}{\sin \xi} = \frac{d\omega}{-2 \cos \xi} = \frac{d\omega^{'}}{2 \sin \xi - \cos \xi \omega^{'}},$$

From $\frac{d\xi}{\sin \xi} = \frac{d\omega}{-2 \cos \xi}$, we have

$$\sin \xi d\omega = -2 \cos \xi d\xi \Longrightarrow d\omega = -2 \cot \xi d\xi \Longrightarrow 2 \ln \sin \xi = -\omega + c,$$

$$\Longrightarrow \sin^2 \xi = c e^{-\omega} \Longrightarrow e^{\omega} \sin^2 \xi = c,$$

and from $\frac{d\xi}{\sin \xi} = \frac{d\omega^{'}}{2 \sin \xi - \cos \xi \omega^{'}}$, we get

$$\sin \xi d\omega^{'} = \left(2 \sin \xi - \cos \xi \omega^{'}\right) d\xi \Longrightarrow \frac{d\omega^{'}}{d\xi} = 2 - \cot \xi \omega^{'},$$

$$\Longrightarrow \frac{d\omega^{'}}{d\xi} + \cot \xi \omega^{'} = 2, \tag{3.341}$$

The integrating factor of equation (3.341) is

$$e^{\int \cot \xi d\xi} = e^{\ln \sin \xi} = \sin \xi,$$

So, equation (3.341) becomes

$$\sin \xi \frac{d\omega^{'}}{d\xi} + \cos \xi \omega^{'} = 2 \sin \xi, \tag{3.342}$$

$$\Longrightarrow \frac{d}{d\xi}\left(\sin \xi \omega^{'}\right) = 2 \sin \xi, \tag{3.343}$$

Integrating equation (3.343) w.r.t ξ, we get

$$\sin \xi \omega^{'} = -2 \cos \xi + c,$$

$$\Longrightarrow \sin \xi \omega^{'} + 2 \cos \xi = c, \tag{3.344}$$

So, the invariants are

$$r = e^{\omega} \sin^2 \xi, \tag{3.345}$$

$$Y = \sin\xi\omega' + 2\cos\xi, \tag{3.346}$$

From equation (3.346), we get

$$\frac{Y}{\sin\xi} = \omega' + 2\cot\xi \Longrightarrow \omega' = \csc\xi Y - 2\cot\xi, \tag{3.347}$$

$$\Longrightarrow \omega'' = -\csc\xi\cot\xi Y + \csc\xi\frac{dY}{d\xi} + 2\csc^2\xi, \tag{3.348}$$

Substituting in equation (3.337), gives

$$-\csc\xi\cot\xi Y + \csc\xi\frac{dY}{d\xi} + 2\csc^2\xi + \cot\xi\csc\xi Y - 2\cot^2\xi = e^\omega + 2,$$

$$\Longrightarrow \csc\xi\frac{dY}{d\xi} + 2\left(\csc^2\xi - \cot^2\xi\right) = e^\omega + 2 \Longrightarrow \frac{dY}{d\xi} = \sin\xi e^\omega,$$

$$\Longrightarrow \frac{dY}{d\xi} = r\csc\xi. \tag{3.349}$$

From equation (3.345), we have

$$\frac{dr}{d\xi} = e^\omega\sin^2\xi\omega' + 2e^\omega\sin\xi\cos\xi \Longrightarrow \frac{dr}{d\xi} = e^\omega\sin^2\xi\left(\csc\xi Y - 2\cot\xi\right) + 2e^\omega\sin^2\xi\cot\xi,$$

$$\Longrightarrow \frac{dr}{d\xi} = r\csc\xi Y - 2r\cot\xi + 2r\cot\xi,$$

$$\Longrightarrow \frac{dr}{d\xi} = r\csc\xi Y, \tag{3.350}$$

Divide equation (3.349) by equation (3.350)

$$\frac{\frac{dY}{d\xi}}{\frac{dr}{d\xi}} = \frac{r\csc\xi}{r\csc\xi Y} \Longrightarrow \frac{dY}{dr} = \frac{1}{Y} \Longrightarrow YdY = dr \Longrightarrow \frac{1}{2}Y^2 = r + c,$$

$$\Longrightarrow Y^2 = 2r + k, \tag{3.351}$$

Back to ξ and ω,

$$\left(\sin\xi\omega' + 2\cos\xi\right)^2 = 2e^\omega\sin^2\xi + k,$$

$$\Longrightarrow \sin\xi\omega' + 2\cos\xi = \pm\sqrt{2e^\omega\sin^2\xi + k}, \tag{3.352}$$

Equation (3.352) has the symmetry

$$X = \sin\xi\frac{\partial}{\partial\xi} - 2\cos\xi\frac{\partial}{\partial\omega}, \tag{3.353}$$

From $\frac{d\xi}{\sin\xi} = \frac{d\omega}{-2\cos\xi}$, we have

$$\sin\xi d\omega = -2\cos\xi d\xi \Longrightarrow d\omega = -2\cot\xi d\xi \Longrightarrow 2\ln\sin\xi = -\omega + c,$$

$$\Longrightarrow \sin^2\xi = ce^{-\omega} \Longrightarrow e^\omega\sin^2\xi = c,$$

Let

$$r = e^\omega\sin^2\xi, \tag{3.354}$$

So

$$\frac{dr}{d\xi} = e^\omega\sin^2\xi\omega' + 2e^\omega\sin\xi\cos\xi \Longrightarrow \frac{dr}{d\xi} = e^\omega\sin\xi\left(\sin\xi\omega' + 2\cos\xi\right),$$

$$\Longrightarrow \frac{dr}{d\xi} = \frac{e^\omega\sin^2\xi}{\sin\xi}\left(\pm\sqrt{2e^\omega\sin^2\xi + k}\right) \Longrightarrow \frac{dr}{d\xi} = \pm r\csc\xi\sqrt{2r + k},$$

$$\Longrightarrow \frac{dr}{r\sqrt{2r+k}} = \pm\csc\xi d\xi, \tag{3.355}$$

Equation (3.355) can be studied in three different cases:

Case 1: $k = 0$,

So, equation (3.355) becomes

$$\frac{dr}{r\sqrt{2r}} = \pm\csc\xi d\xi, \tag{3.356}$$

$$\Longrightarrow \frac{1}{\sqrt{2}} \frac{dr}{r^{\frac{3}{2}}} = \pm \csc \xi d\xi, \tag{3.357}$$

Integrating equation (3.357), get

$$\frac{-\sqrt{2}}{\sqrt{r}} = \pm \ln (\csc \xi + \cot \xi) + c,$$

$$\Longrightarrow r = \left[\frac{2}{(\ln(\csc \xi + \cot \xi) + c)^2}\right], \tag{3.358}$$

$$\Longrightarrow e^{\omega} \sin^2 \xi = \left[\frac{2}{(\ln(\csc \xi + \cot \xi) + c)^2}\right], \tag{3.359}$$

$$\Longrightarrow \omega = \ln \left[\frac{2\csc^2 \xi}{(\ln(\csc \xi + \cot \xi) + c)^2}\right], \tag{3.360}$$

Back to q,v and P,

$$P = \ln \left[\frac{2\csc^2 q}{(\ln(\csc q + \cot q) + c)^2}\right], \tag{3.361}$$

Back to x,y,t and u,

$$u(x, y, t) = \ln \left[\frac{2\csc^2 x}{(\ln(\csc x + \cot x) + c)^2}\right],$$

$$\Longrightarrow u(x, y, t) = \ln 2 - 2\ln|\sin x| - 2\ln|\ln(\csc x + \cot x) + c|. \tag{3.362}$$

Case 2: $k > 0$,

Integrating equation (3.355), we get

$$\frac{-2}{\sqrt{k}} \tanh^{-1}\left(\frac{\sqrt{2r+k}}{\sqrt{k}}\right) = \pm \ln (\csc \xi + \cot \xi) + c,$$

$$\Longrightarrow r = \frac{k}{2}\left(\tanh^2\left[\frac{\sqrt{k}}{2} \ln (\csc \xi + \cot \xi) + c\right] - 1\right), \tag{3.363}$$

$$\Longrightarrow e^{\omega} \sin^2 \xi = \frac{k}{2}\left(\tanh^2\left[\frac{\sqrt{k}}{2} \ln (\csc \xi + \cot \xi) + c\right] - 1\right), \tag{3.364}$$

$$\Longrightarrow \omega = \ln \left|\frac{k}{2} \csc^2 \xi \sec h^2 \left[\frac{\sqrt{k}}{2} \ln (\csc \xi + \cot \xi) + c\right]\right|, \tag{3.365}$$

Back to q,v and P,

$$P = \ln \left|\frac{k}{2} \csc^2 q \sec h^2 \left[\frac{\sqrt{k}}{2} \ln (\csc q + \cot q) + c\right]\right|, \tag{3.366}$$

Back to x,y,t and u,

$$u(x, y, t) = \ln \left|\frac{k}{2} \csc^2 x \sec h^2 \left[\frac{\sqrt{k}}{2} \ln (\csc x + \cot x) + c\right]\right|. \tag{3.367}$$

Remark:

To integrate the function $\frac{1}{r\sqrt{2r+k}}$,

we let $u = 2r + k \Longrightarrow du = 2dr$,

and $r = \frac{1}{2}(u - k)$,

So

$$\int \frac{dr}{r\sqrt{2r+k}} = \int \frac{du}{(u-k)u^{\frac{1}{2}}},$$

Now, let $t = u^{\frac{1}{2}} \Longrightarrow dt = \frac{1}{2u^{\frac{1}{2}}} du$,

and $t^2 = u$,

So

$$\int \frac{du}{(u-k)u^{\frac{1}{2}}} = 2\int \frac{dt}{t^2 - \left(\sqrt{k}\right)^2},$$

Now, let $t = \sec\theta \Longrightarrow dt = \sec\theta \tan\theta d\theta$,

and using $\sec^2\theta - 1 = \tan^2\theta$, we get

So

$$\int \frac{dt}{t^2-\left(\sqrt{k}\right)^2} = \frac{-2}{\sqrt{k}} \tanh^{-1}\left(\frac{t}{\sqrt{k}}\right),$$
$$= \frac{-2}{\sqrt{k}} \tanh^{-1}\left(\frac{\sqrt{u}}{\sqrt{k}}\right),$$
$$= \frac{-2}{\sqrt{k}} \tanh^{-1}\left(\frac{\sqrt{2r+k}}{\sqrt{k}}\right).$$

Case 3: $k < 0$,

Integrating equation (3.355), we get

$$\frac{2}{\sqrt{-k}} \tan^{-1}\left(\frac{\sqrt{2r+k}}{\sqrt{-k}}\right) = \pm \ln\left(\csc\xi + \cot\xi\right) + c,$$
$$\Longrightarrow r = \frac{-k}{2}\left(\tan^2\left[\frac{\sqrt{-k}}{2}\ln\left(\csc\xi + \cot\xi\right) + c\right] + 1\right), \tag{3.368}$$
$$\Longrightarrow e^{\omega}\sin^2\xi = \frac{-k}{2}\sec^2\left[\frac{\sqrt{-k}}{2}\ln\left(\csc\xi + \cot\xi\right) + c\right], \tag{3.369}$$
$$\Longrightarrow \omega = \ln\left|\frac{-k}{2}\csc^2\xi\sec^2\left[\frac{\sqrt{-k}}{2}\ln\left(\csc\xi + \cot\xi\right) + c\right]\right|, \tag{3.370}$$

Back to q,v and P,

$$P = \ln\left|\frac{-k}{2}\csc^2 q\sec^2\left[\frac{\sqrt{-k}}{2}\ln\left(\csc q + \cot q\right) + c\right]\right|, \tag{3.371}$$

Back to x,y,t and u,

$$u\left(x, y, t\right) = \ln\left|\frac{-k}{2}\csc^2 x\sec^2\left[\frac{\sqrt{-k}}{2}\ln\left(\csc x + \cot x\right) + c\right]\right|. \tag{3.372}$$

3.2.2.2 Reduction Using $\langle X_3, X_1\rangle$

Using section 3.2.1.3, equation (3.336) reduces to

$$\left(1-\xi^2\right)\omega^{''} - 2\xi\omega^{'} - e^{\omega} - 2 = 0. \tag{3.373}$$

Equation (3.373) has the symmetry

$$X = \left(\xi^2 - 1\right)\frac{\partial}{\partial\xi} - 2\xi\frac{\partial}{\partial\omega}, \tag{3.374}$$

The first prolongation of X is

$$X^{[1]} = \left(\xi^2 - 1\right)\frac{\partial}{\partial\xi} - 2\xi\frac{\partial}{\partial\omega} + \left(-2 - 2\xi\omega^{'}\right)\frac{\partial}{\partial\omega^{'}}, \tag{3.375}$$

The characteristic system

$$\frac{d\xi}{\xi^2-1} = \frac{d\omega}{-2\xi} = \frac{d\omega^{'}}{-2-2\xi\omega^{'}},$$

From $\frac{d\xi}{\xi^2-1} = \frac{d\omega}{-2\xi}$, we have

$$\left(\xi^2 - 1\right)d\omega = -2\xi d\xi \Longrightarrow d\omega = \frac{-2\xi}{\left(\xi^2-1\right)}d\xi \Longrightarrow \ln\left(\xi^2 - 1\right) = -\omega + c,$$
$$\Longrightarrow e^{\omega}\left(\xi^2 - 1\right) = c, \tag{3.376}$$

and from $\frac{d\xi}{\xi^2-1} = \frac{d\omega^{'}}{-2-2\xi\omega^{'}}$, we get

$$\left(\xi^2 - 1\right)d\omega^{'} = \left(-2 - 2\xi\omega^{'}\right)d\xi \Longrightarrow \frac{d\omega^{'}}{d\xi} = \frac{-2-2\xi\omega^{'}}{\left(\xi^2-1\right)} \Longrightarrow \frac{d\omega^{'}}{d\xi} = \frac{-2}{\xi^2-1} - \frac{2\xi\omega^{'}}{\xi^2-1},$$
$$\Longrightarrow \frac{d\omega^{'}}{d\xi} + \frac{2\xi}{\xi^2-1}\omega^{'} = \frac{-2}{\xi^2-1}, \tag{3.377}$$

The integrating factor of equation (3.377) is

$$e^{\int\frac{2\xi}{\xi^2-1}d\xi} = e^{\ln\left(\xi^2-1\right)} = \xi^2 - 1,$$

So, equation (3.377) becomes

$$\left(\xi^2-1\right)\frac{d\omega^{'}}{d\xi}+2\xi\omega^{'}=-2,$$
$$\Longrightarrow \frac{d}{d\xi}\left(\left(\xi^2-1\right)\omega^{'}\right)=-2, \tag{3.378}$$

Integrating equation (3.378) w.r.t ξ, we get

$$\left(\xi^2-1\right)\omega^{'}=-2\xi+c,$$
$$\Longrightarrow \left(\xi^2-1\right)\omega^{'}+2\xi=c, \tag{3.379}$$

So, the invariants are

$$r=e^{\omega}\left(\xi^2-1\right), \tag{3.380}$$
$$Y=\left(\xi^2-1\right)\omega^{'}+2\xi, \tag{3.381}$$

From equation (3.381), we get

$$\left(\xi^2-1\right)\omega^{'}=Y-2\xi \Longrightarrow \omega^{'}=\frac{1}{\xi^2-1}Y-\frac{2\xi}{\xi^2-1}, \tag{3.382}$$
$$\Longrightarrow \omega^{''}=\frac{1}{\xi^2-1}\frac{dY}{d\xi}-\frac{2\xi}{(\xi^2-1)^2}Y+\frac{2\xi^2+2}{(\xi^2-1)^2}, \tag{3.383}$$

Substitute in equation (3.373)

$$-\frac{dY}{d\xi}+\frac{2\xi}{(\xi^2-1)}Y-\frac{2\xi^2+2}{(\xi^2-1)}-\frac{2\xi}{\xi^2-1}Y+\frac{4\xi^2}{\xi^2-1}=e^{\omega}+2,$$
$$\Longrightarrow -\frac{dY}{d\xi}+2=e^{\omega}+2,$$
$$\Longrightarrow \frac{dY}{d\xi}=\frac{-r}{\xi^2-1}, \tag{3.384}$$

From equation (3.380), we have

$$\frac{dr}{d\xi}=e^{\omega}\left(\xi^2-1\right)\omega^{'}+2\xi e^{\omega} \Longrightarrow \frac{dr}{d\xi}=e^{\omega}\left(\left(\xi^2-1\right)\omega^{'}+2\xi\right),$$
$$\Longrightarrow \frac{dr}{d\xi}=e^{\omega}Y,$$
$$\Longrightarrow \frac{dr}{d\xi}=\frac{r}{\xi^2-1}Y, \tag{3.385}$$

Divide equation (3.384) by equation (3.385)

$$\frac{\frac{dY}{d\xi}}{\frac{dr}{d\xi}}=\frac{\frac{-r}{\xi^2-1}}{\frac{r}{\xi^2-1}Y} \Longrightarrow \frac{dY}{dr}=\frac{-1}{Y} \Longrightarrow YdY=-dr \Longrightarrow \frac{1}{2}Y^2=-r+c,$$
$$\Longrightarrow Y^2=-2r+k, \tag{3.386}$$

Back to ξ and ω,

$$\left(\left(\xi^2-1\right)\omega^{'}+2\xi\right)^2=-2e^{\omega}\left(\xi^2-1\right)+k,$$
$$\Longrightarrow \left(\xi^2-1\right)\omega^{'}+2\xi=\pm\sqrt{-2r+k}, \tag{3.387}$$

Equation (3.387) has the symmetry

$$X=\left(\xi^2-1\right)\frac{\partial}{\partial\xi}-2\xi\frac{\partial}{\partial\omega}, \tag{3.388}$$

From $\frac{d\xi}{\xi^2-1}=\frac{d\omega}{-2\xi}$, we have

$$\left(\xi^2-1\right)d\omega=-2\xi d\xi \Longrightarrow d\omega=\frac{-2\xi}{(\xi^2-1)}d\xi \Longrightarrow \ln\left(\xi^2-1\right)=-\omega+c,$$
$$\Longrightarrow e^{\omega}\left(\xi^2-1\right)=c,$$

Let

$$r=e^{\omega}\left(\xi^2-1\right), \tag{3.389}$$

So

$$\frac{dr}{d\xi} = e^{\omega}\left(\xi^2-1\right)\omega' + 2\xi e^{\omega} \Longrightarrow \frac{dr}{d\xi} = e^{\omega}\left(\left(\xi^2-1\right)\omega' + 2\xi\right),$$
$$\Longrightarrow \frac{dr}{d\xi} = \frac{e^{\omega}\left(\xi^2-1\right)}{\left(\xi^2-1\right)}\left(\left(\xi^2-1\right)\omega' + 2\xi\right) \Longrightarrow \frac{dr}{d\xi} = \pm\frac{r}{\left(\xi^2-1\right)}\sqrt{-2r+k},$$
$$\Longrightarrow \frac{dr}{r\sqrt{-2r+k}} = \pm\frac{d\xi}{\xi^2-1}, \tag{3.390}$$

Equation (3.390) can be studied in three different cases:

Case 1: $k=0$,

So, equation (3.390) becomes

$$\frac{dr}{r\sqrt{-2r}} = \pm\frac{d\xi}{\xi^2-1}, \tag{3.391}$$

Integrating equation (3.391), we get

$$\frac{-2}{\sqrt{-2r}} = \pm\tanh^{-1}(\xi)+c,$$
$$\Longrightarrow r = \frac{-2}{\left(\tanh^{-1}(\xi)+c\right)^2}, \tag{3.392}$$
$$\Longrightarrow e^{\omega}\left(\xi^2-1\right) = \frac{-2}{\left(\tanh^{-1}(\xi)+c\right)^2}, \tag{3.393}$$

Back to q,v and P,

$$e^{P}\left(\sin^2 q\sin^2 v-1\right) = \frac{-2}{\left(\tanh^{-1}(\sin q\sin v)+c\right)^2},$$
$$\Longrightarrow e^{P} = \frac{2}{\left(\tanh^{-1}(\sin q\sin v)+c\right)^2\left(1-\sin^2 q\sin^2 v\right)},$$
$$\Longrightarrow P = \ln 2-\ln\left(1-\sin^2 q\sin^2 v\right)-2\ln\left[\tanh^{-1}(\sin q\sin v)+c\right], \tag{3.394}$$

Back to x,y,t and u,

$$u(x,y,t) = \ln 2-\ln\left(1-\sin^2 x\sin^2 y\right)-2\ln\left[\tanh^{-1}(\sin x\sin y)+c\right]. \tag{3.395}$$

Remark:

To integrate the function $\frac{1}{r\sqrt{-2r}}$,

we let $u=-2r \Longrightarrow du=-2dr$,

and $r=\frac{-1}{2}u$,

So

$$\int\frac{dr}{r\sqrt{-2r}} = \int\frac{du}{u^{\frac{3}{2}}} = \frac{-2}{\sqrt{u}},$$
$$= \frac{-2}{\sqrt{-2r}}.$$

Case 2: $k>0$,

Integrating equation (3.390), we get

$$\frac{-2}{\sqrt{k}}\tanh^{-1}\left(\frac{\sqrt{-2r+k}}{\sqrt{k}}\right) = \pm\tanh^{-1}(\xi)+c,$$
$$\Longrightarrow r = \frac{-k}{2}\left(\tanh^2\left[\frac{\sqrt{k}}{2}\tanh^{-1}(\xi)+c\right]-1\right), \tag{3.396}$$
$$\Longrightarrow e^{\omega}\left(\xi^2-1\right) = \frac{-k}{2}\left(\tanh^2\left[\frac{\sqrt{k}}{2}\tanh^{-1}(\xi)+c\right]-1\right), \tag{3.397}$$
$$\Longrightarrow \omega = \ln\left|\frac{k}{2(1-\xi^2)}\sec h^2\left[\frac{\sqrt{k}}{2}\tanh^{-1}(\xi)+c\right]\right|, \tag{3.398}$$

Back to q,v and P,

$$P = \ln\left|\frac{k}{2(1-\sin^2 q\sin^2 v)}\sec h^2\left[\frac{\sqrt{k}}{2}\tanh^{-1}(\sin q\sin v)+c\right]\right|, \tag{3.399}$$

Back to x,y,t and u

$$u(x,y,t)=\ln\left|\frac{k}{2(1-\sin^2 x\sin^2 y)}\sec h^2\left[\frac{\sqrt{k}}{2}\tanh^{-1}(\sin x\sin y)+c\right]\right|. \tag{3.400}$$

Case 3: $k<0$,

Integrating equation (3.390), we get

$$\frac{2}{\sqrt{-k}}\tan^{-1}\left(\frac{\sqrt{-2r+k}}{\sqrt{-k}}\right)=\pm\tanh^{-1}(\xi)+c,$$

$$\Longrightarrow r=\frac{k}{2}\left(\tan^2\left[\frac{\sqrt{-k}}{2}\tanh^{-1}(\xi)+c\right]+1\right), \tag{3.401}$$

$$\Longrightarrow e^{\omega}\left(\xi^2-1\right)=\frac{k}{2}\sec^2\left[\frac{\sqrt{-k}}{2}\tanh^{-1}(\xi)+c\right], \tag{3.402}$$

$$\Longrightarrow \omega=\ln\left|\frac{k}{2(\xi^2-1)}\sec^2\left[\frac{\sqrt{-k}}{2}\tanh^{-1}(\xi)+c\right]\right|, \tag{3.403}$$

Back to q,v and P,

$$P=\ln\left|\frac{k}{2(\sin^2 q\sin^2 v-1)}\sec^2\left[\frac{\sqrt{-k}}{2}\tanh^{-1}(\sin q\sin v)+c\right]\right|, \tag{3.404}$$

Back to x,y,t and u,

$$u(x,y,t)=\ln\left|\frac{k}{2(\sin^2 x\sin^2 y-1)}\sec^2\left[\frac{\sqrt{-k}}{2}\tanh^{-1}(\sin x\sin y)+c\right]\right|. \tag{3.405}$$

3.2.2.3 Reduction Using $\langle X_3, X_2\rangle$

Using section 3.2.1.4, equation (3.336) reduces to

$$\left(1-\xi^2\right)\omega''-2\xi\omega'-e^{\omega}-2=0. \tag{3.406}$$

Equation (3.406) has the symmetry

$$X=\left(\xi^2-1\right)\frac{\partial}{\partial\xi}-2\xi\frac{\partial}{\partial\omega}, \tag{3.407}$$

The first prolongation of X is

$$X^{[1]}=\left(\xi^2-1\right)\frac{\partial}{\partial\xi}-2\xi\frac{\partial}{\partial\omega}+\left(-2-2\xi\omega'\right)\frac{\partial}{\partial\omega'}, \tag{3.408}$$

The characteristic system

$$\frac{d\xi}{\xi^2-1}=\frac{d\omega}{-2\xi}=\frac{d\omega'}{-2-2\xi\omega'},$$

From $\frac{d\xi}{\xi^2-1}=\frac{d\omega}{-2\xi}$, we have

$$\left(\xi^2-1\right)d\omega=-2\xi d\xi\Longrightarrow d\omega=\frac{-2\xi}{(\xi^2-1)}d\xi\Longrightarrow\ln\left(\xi^2-1\right)=-\omega+c,$$

$$\Longrightarrow e^{\omega}\left(\xi^2-1\right)=c, \tag{3.409}$$

and from $\frac{d\xi}{\xi^2-1}=\frac{d\omega'}{-2-2\xi\omega'}$, we get

$$\left(\xi^2-1\right)d\omega'=\left(-2-2\xi\omega'\right)d\xi\Longrightarrow\frac{d\omega'}{d\xi}=\frac{-2-2\xi\omega'}{(\xi^2-1)}\Longrightarrow\frac{d\omega'}{d\xi}=\frac{-2}{\xi^2-1}-\frac{2\xi\omega'}{\xi^2-1},$$

$$\Longrightarrow\frac{d\omega'}{d\xi}+\frac{2\xi}{\xi^2-1}\omega'=\frac{-2}{\xi^2-1}, \tag{3.410}$$

The integrating factor of equation (3.410) is

$$e^{\int\frac{2\xi}{\xi^2-1}d\xi}=e^{\ln\left(\xi^2-1\right)}=\xi^2-1,$$

So, equation (3.410) becomes

$$\left(\xi^2-1\right)\frac{d\omega'}{d\xi}+2\xi\omega'=-2,$$

$$\Longrightarrow\frac{d}{d\xi}\left(\left(\xi^2-1\right)\omega'\right)=-2, \tag{3.411}$$

Integrating equation (3.411) w.r.t ξ, we get

$$\left(\xi^2-1\right)\omega^{'}=-2\xi+c,$$
$$\Longrightarrow \left(\xi^2-1\right)\omega^{'}+2\xi=c, \tag{3.412}$$

So, the invariants are

$$r=e^{\omega}\left(\xi^2-1\right), \tag{3.413}$$
$$Y=\left(\xi^2-1\right)\omega^{'}+2\xi, \tag{3.414}$$

From equation (3.414), we get

$$\left(\xi^2-1\right)\omega^{'}=Y-2\xi,$$
$$\Longrightarrow \omega^{'}=\tfrac{1}{\xi^2-1}Y-\tfrac{2\xi}{\xi^2-1}, \tag{3.415}$$
$$\Longrightarrow \omega^{''}=\tfrac{1}{\xi^2-1}\tfrac{dY}{d\xi}-\tfrac{2\xi}{(\xi^2-1)^2}Y+\tfrac{2\xi^2+2}{(\xi^2-1)^2}, \tag{3.416}$$

Substitute in equation (3.406)

$$-\tfrac{dY}{d\xi}+\tfrac{2\xi}{(\xi^2-1)}Y-\tfrac{2\xi^2+2}{(\xi^2-1)}-\tfrac{2\xi}{\xi^2-1}Y+\tfrac{4\xi^2}{\xi^2-1}=e^{\omega}+2,$$
$$\Longrightarrow -\tfrac{dY}{d\xi}+2=e^{\omega}+2,$$
$$\Longrightarrow \tfrac{dY}{d\xi}=\tfrac{-r}{\xi^2-1}, \tag{3.417}$$

From equation (3.413), we have

$$\tfrac{dr}{d\xi}=e^{\omega}\left(\xi^2-1\right)\omega^{'}+2\xi e^{\omega}\Longrightarrow \tfrac{dr}{d\xi}=e^{\omega}\left(\left(\xi^2-1\right)\omega^{'}+2\xi\right),$$
$$\Longrightarrow \tfrac{dr}{d\xi}=e^{\omega}Y,$$
$$\Longrightarrow \tfrac{dr}{d\xi}=\tfrac{r}{\xi^2-1}Y, \tag{3.418}$$

Divide equation (3.417) by equation (3.418)

$$\frac{\frac{dY}{d\xi}}{\frac{dr}{d\xi}}=\frac{\frac{-r}{\xi^2-1}}{\frac{r}{\xi^2-1}Y}\Longrightarrow \tfrac{dY}{dr}=\tfrac{-1}{Y}\Longrightarrow YdY=-dr\Longrightarrow \tfrac{1}{2}Y^2=-r+c,$$
$$\Longrightarrow Y^2=-2r+k, \tag{3.419}$$

Back to ξ and ω,

$$\left(\left(\xi^2-1\right)\omega^{'}+2\xi\right)^2=-2e^{\omega}\left(\xi^2-1\right)+k,$$
$$\Longrightarrow \left(\xi^2-1\right)\omega^{'}+2\xi=\pm\sqrt{-2r+k}, \tag{3.420}$$

Equation (3.420) has the symmetry

$$X=\left(\xi^2-1\right)\tfrac{\partial}{\partial\xi}-2\xi\tfrac{\partial}{\partial\omega}, \tag{3.421}$$

From $\frac{d\xi}{\xi^2-1}=\frac{d\omega}{-2\xi}$, we have

$$\left(\xi^2-1\right)d\omega=-2\xi d\xi\Longrightarrow d\omega=\tfrac{-2\xi}{(\xi^2-1)}d\xi\Longrightarrow \ln\left(\xi^2-1\right)=-\omega+c,$$
$$\Longrightarrow e^{\omega}\left(\xi^2-1\right)=c,$$

Let

$$r=e^{\omega}\left(\xi^2-1\right), \tag{3.422}$$

So

$$\tfrac{dr}{d\xi}=e^{\omega}\left(\xi^2-1\right)\omega^{'}+2\xi e^{\omega}\Longrightarrow \tfrac{dr}{d\xi}=e^{\omega}\left(\left(\xi^2-1\right)\omega^{'}+2\xi\right),$$
$$\Longrightarrow \tfrac{dr}{d\xi}=\tfrac{e^{\omega}\left(\xi^2-1\right)}{(\xi^2-1)}\left(\left(\xi^2-1\right)\omega^{'}+2\xi\right)\Longrightarrow \tfrac{dr}{d\xi}=\pm\tfrac{r}{(\xi^2-1)}\sqrt{-2r+k},$$

$$\Longrightarrow \frac{dr}{r\sqrt{-2r+k}} = \pm\frac{d\xi}{\xi^2-1}, \tag{3.423}$$

Equation (3.423) can be studied in three different cases:

Case 1: $k = 0$,

So, equation (3.423) becomes

$$\frac{dr}{r\sqrt{-2r}} = \pm\frac{d\xi}{\xi^2-1}, \tag{3.424}$$

Integrating equation (3.424), we get

$$\frac{-2}{\sqrt{-2r}} = \pm\tanh^{-1}(\xi) + c,$$

$$\Longrightarrow r = \frac{-2}{\left(\tanh^{-1}(\xi)+c\right)^2}, \tag{3.425}$$

$$\Longrightarrow e^{\omega}\left(\xi^2-1\right) = \frac{-2}{\left(\tanh^{-1}(\xi)+c\right)^2}, \tag{3.426}$$

Back to q,v and P,

$$e^{P}\left(\sin^2 q\cos^2 v - 1\right) = \frac{-2}{\left(\tanh^{-1}(\sin q\cos v)+c\right)^2},$$

$$\Longrightarrow e^{P} = \frac{2}{\left(\tanh^{-1}(\sin q\cos v)+c\right)^2\left(1-\sin^2 q\cos^2 v\right)},$$

$$\Longrightarrow P = \ln 2 - \ln\left(1-\sin^2 q\cos^2 v\right) - 2\ln\left[\tanh^{-1}(\sin q\cos v)+c\right], \tag{3.427}$$

Back to x,y,t and u,

$$u(x,y,t) = \ln 2 - \ln\left(1-\sin^2 x\cos^2 y\right) - 2\ln\left[\tanh^{-1}(\sin x\cos y)+c\right]. \tag{3.428}$$

Case 2: $k > 0$,

Integrating equation (3.423), we get

$$\frac{-2}{\sqrt{k}}\tanh^{-1}\left(\frac{\sqrt{-2r+k}}{\sqrt{k}}\right) = \pm\tanh^{-1}(\xi) + c,$$

$$\Longrightarrow r = \frac{-k}{2}\left(\tanh^2\left[\frac{\sqrt{k}}{2}\tanh^{-1}(\xi)+c\right]-1\right), \tag{3.429}$$

$$\Longrightarrow e^{\omega}\left(\xi^2-1\right) = \frac{-k}{2}\left(\tanh^2\left[\frac{\sqrt{k}}{2}\tanh^{-1}(\xi)+c\right]-1\right), \tag{3.430}$$

$$\Longrightarrow \omega = \ln\left|\frac{k}{2(1-\xi^2)}\sec h^2\left[\frac{\sqrt{k}}{2}\tanh^{-1}(\xi)+c\right]\right|, \tag{3.431}$$

Back to q,v and P,

$$P = \ln\left|\frac{k}{2(1-\sin^2 q\cos^2 v)}\sec h^2\left[\frac{\sqrt{k}}{2}\tanh^{-1}(\sin q\cos v)+c\right]\right|, \tag{3.432}$$

Back to x,y,t and u,

$$u(x,y,t) = \ln\left|\frac{k}{2(1-\sin^2 x\cos^2 y)}\sec h^2\left[\frac{\sqrt{k}}{2}\tanh^{-1}(\sin x\cos y)+c\right]\right|. \tag{3.433}$$

Case 3: $k < 0$,

Integrating equation (3.423), we get

$$\frac{2}{\sqrt{-k}}\tan^{-1}\left(\frac{\sqrt{-2r+k}}{\sqrt{-k}}\right) = \pm\tanh^{-1}(\xi) + c,$$

$$\Longrightarrow r = \frac{k}{2}\left(\tan^2\left[\frac{\sqrt{-k}}{2}\tanh^{-1}(\xi)+c\right]+1\right), \tag{3.434}$$

$$\Longrightarrow e^{\omega}\left(\xi^2-1\right) = \frac{k}{2}\sec^2\left[\frac{\sqrt{-k}}{2}\tanh^{-1}(\xi)+c\right], \tag{3.435}$$

$$\Longrightarrow \omega = \ln\left|\frac{k}{2(\xi^2-1)}\sec^2\left[\frac{\sqrt{-k}}{2}\tanh^{-1}(\xi)+c\right]\right|, \tag{3.436}$$

Back to q,v and P,

$$P = \ln\left|\frac{k}{2(\sin^2 q\cos^2 v-1)}\sec^2\left[\frac{\sqrt{-k}}{2}\tanh^{-1}(\sin q\cos v)+c\right]\right|, \tag{3.437}$$

Back to x,y,t and u,

$$u\left(x,y,t\right)=\ln\left|\tfrac{k}{2(\sin^2 x\cos^2 y-1)}\sec^2\left[\tfrac{\sqrt{-k}}{2}\tanh^{-1}\left(\sin x\cos y\right)+c\right]\right|. \tag{3.438}$$

CHAPTER 4
Symmetry Analysis and reductions of linear Klein Gordon equation on sphere

In this chapter, we will study the linear Klein Gordon equation of the form

$$u_{tt} = \triangle u + f(u), \tag{4.1}$$

where $f(u)$ is a linear function, i.e. $f(u) = au$, where $a \neq 0$.

So, we will study the following equation

$$u_{tt} = u_{xx} + (\cot x)\, u_x + \left(\csc^2 x\right) u_{yy} + au. \tag{4.2}$$

We will have the same determining equation as obtained in section 3.1 with $f(u) = au$.

So, the system of determining equations is

$$e_1 : \xi_u = 0,$$

$$e_2 : \theta_u = 0,$$

$$e_3 : \csc^2 x \xi_y + \theta_x = 0,$$

$$e_4 : \tau_u = 0,$$

$$e_5 : \xi_x - \tau_t = 0,$$

$$e_6 : \cot x \xi - \tau_t + \theta_y = 0,$$

$$e_7 : \csc^2 x \tau_y - \theta_t = 0,$$

$$e_8 : \tau_x - \xi_t = 0,$$

$$e_9 : \phi_{uu} = 0,$$

$$e_{10} : \cot x \tau_x - \tau_{tt} + \tau_{xx} + \csc^2 x \tau_{yy} + 2\phi_{tu} = 0,$$

$$e_{11} : \cot x \theta_x - \theta_{tt} + \theta_{xx} + \csc^2 x \theta_{yy} - 2 \csc^2 x \phi_{yu} = 0,$$

$$e_{12} : -\cot x \xi_x - \xi_{tt} + \xi_{xx} + \csc^2 x \xi + \csc^2 x \xi_{yy} - 2\phi_{xu} = 0,$$

$$e_{13} : au\phi_u - a\phi - \phi_{xx} + \phi_{tt} - 2au\tau_t - \csc^2 x \phi_{yy} - \cot x \phi_x = 0.$$

4.1 Symmetry Algebra

In this section, we will try to simplify the determining equations in order to find the symmetry algebra of equation (4.2).

From $(e_9) : \phi_{uu} = 0 \Longrightarrow \phi(x, y, t, u) = A(x, y, t)\, u + B(x, y, t)\,, \quad (4.3)$

$$\Longrightarrow \phi_u = A(x, y, t)\,. \tag{4.4}$$

Putting in equation (e_{13}), gives

$$auA - a(Au + B) - (A_{xx}u + B_{xx}) + (A_{tt}u + B_{tt}) - 2au\tau_t - \csc^2 x (A_{yy}u + B_{yy})$$

$$- \cot x (A_x u + B_x) = 0,$$

$$\Longrightarrow \left[A_{tt} - A_{xx} - 2a\tau_t - \csc^2 x A_{yy} - \cot x A_x\right] u$$
$$+ \left[B_{tt} - B_{xx} - \csc^2 x B_{yy} - \cot x B_x - aB\right] = 0, \tag{4.5}$$

Differentiating equation (4.5) with respect to u, gives

$$\Longrightarrow A_{tt} - A_{xx} - 2a\tau_t - \csc^2 x A_{yy} - \cot x A_x = 0, \tag{4.6}$$

So, equation (4.5) becomes

$$B_{tt} - B_{xx} - \csc^2 x B_{yy} - \cot x B_x - aB = 0, \tag{4.7}$$

which means that B satisfies the original equation.

Next, we look at

$$(3.22) : \tau_{tt} + 2\phi_{tu} = 0 \Longrightarrow \tau_{tt} + 2A_t = 0, \tag{4.8}$$
$$(3.26) : \tau_{yt} - 2\phi_{yu} = 0 \Longrightarrow \tau_{yt} - 2A_y = 0, \tag{4.9}$$
$$(3.28) : \tau_{xt} + 2\phi_{xu} = 0 \Longrightarrow \tau_{xt} + 2A_x = 0, \tag{4.10}$$

Differentiating equation (4.8) with respect to y,

$$\Longrightarrow A_{ty} = 0 \Longrightarrow (A_y)_t = 0, \tag{4.11}$$

Differentiating equation (4.10) with respect to y,

$$\Longrightarrow A_{xy} = 0 \Longrightarrow (A_y)_x = 0, \tag{4.12}$$

From equations (4.11) and (4.12), we get

$$A_y = f(y), \tag{4.13}$$

Differentiating equation (4.6) with respect to y,

$$0 - 0 - 2a\tau_{yt} - \csc^2 x A_{yyy} - 0 = 0 \Longrightarrow \csc^2 x A_{yyy} + 4aA_y = 0,$$
$$\Longrightarrow \csc^2 x f''(y) + 4af(y) = 0, \tag{4.14}$$

Differentiating equation (4.14) with respect to x,

$$-2\csc^2 x \cot x f''(y) = 0 \Longrightarrow f''(y) = 0, \tag{4.15}$$

Putting equation (4.15) in equation (4.14) gives

$$f(y) = 0 \Longrightarrow A_y = 0,$$
$$\Longrightarrow A = A(x,t), \tag{4.16}$$

So, equation (4.9) becomes

$$\tau_{yt} = 0. \tag{4.17}$$

Differentiating equation (4.8) with respect to t,

$$\tau_{ttt} = -2A_{tt}, \tag{4.18}$$

Using equation (3.8), equation (4.18) becomes

$$\tau_{xxt} = -2A_{tt}, \tag{4.19}$$

Differentiating equation (4.10) with respect to x,

$$\tau_{xxt} = -2A_{xx}, \tag{4.20}$$

From equations (4.19) and (4.20) we get

$$A_{xx} = A_{tt}, \tag{4.21}$$

Using equations (4.16) and (4.21) in equation (4.6) gives

$$2a\tau_t + \cot x A_x = 0, \tag{4.22}$$

Differentiating equation (4.22) with respect to t, and using equation (4.8)

$$2a\tau_{tt} + \cot x A_{xt} = 0 \Longrightarrow -4aA_t + \cot x A_{xt} = 0,$$

$$\Longrightarrow \left(-4aA + \cot x A_x\right)_t = 0,$$

$$\Longrightarrow -4aA + \cot x A_x = g\left(x\right). \tag{4.23}$$

Now

$$\left(e_{10}\right)_t = 0 \Longrightarrow \cot x\tau_{xt} + \csc^2 x\tau_{yyt} + 2\phi_{ttu} = 0 \Longrightarrow \cot x\tau_{xt} + 2A_{tt} = 0,$$

$$\Longrightarrow -2\cot x A_x + 2A_{tt} = 0,$$

$$\Longrightarrow A_{tt} = \cot x A_x, \tag{4.24}$$

Putting equation (4.23) in equation (4.24)

$$A_{tt} = g\left(x\right) + 4aA, \tag{4.25}$$

Differentiating equation (4.25) with respect to t,

$$A_{ttt} = 4aA_t \Longrightarrow \left(A_{tt} - 4aA\right)_t = 0,$$

$$\Longrightarrow A_{tt} - 4aA = h\left(x\right). \tag{4.26}$$

Now

$$\left(e_{12}\right)_t = 0 \Longrightarrow -\cot x\xi_{xt} + \csc^2 x\xi_t + \csc^2 x\xi_{yyt} - 2\phi_{xtu} = 0,$$

$$\Longrightarrow -\cot x\xi_{xt} + \csc^2 x\xi_t + \csc^2 x\xi_{yyt} - 2A_{xt} = 0,$$

$$\Longrightarrow -\cot x\tau_{tt} + \csc^2 x\tau_x + \csc^2 x\tau_{xyy} - 2A_{xt} = 0, \tag{4.27}$$

Differentiating equation (4.27) with respect to t,

$$-\cot x\tau_{ttt} + \csc^2 x\tau_{xt} + \csc^2 x\tau_{xyyt} - 2A_{xtt} = 0, \tag{4.28}$$

Using equations (4.8) and (4.10) in equation (4.28) gives

$$\cot x A_{tt} - \csc^2 x A_x - A_{xtt} = 0, \tag{4.29}$$

Putting equation (4.24) in equation (4.29)

$$\cot^2 x A_x - \csc^2 x A_x - A_{xtt} = 0 \Longrightarrow \left(\cot^2 x - \csc^2 x\right) A_x - A_{xtt} = 0,$$

$$\Longrightarrow A_x + A_{xtt} = 0 \Longrightarrow \left(A + A_{tt}\right)_x = 0,$$

$$\Longrightarrow A + A_{tt} = p\left(t\right), \tag{4.30}$$

Using equation (4.26) in equation (4.30)

$$A + 4aA + h\left(x\right) = p\left(t\right) \Longrightarrow \left(4a+1\right) A = p\left(t\right) - h\left(x\right),$$

$$\Longrightarrow A = \tfrac{p(t)-h(x)}{4a+1}, \tag{4.31}$$

Putting equation (4.31) in equation (4.24)

$$\frac{1}{4a+1}p^{''}(t) = \frac{1}{4a+1}\cot x h^{'}(x),$$
$$\Longrightarrow p^{''}(t) = \cot x h^{'}(x), \tag{4.32}$$

which means that

$$p^{''}(t) = 0 \text{ and } h^{'}(x) = 0,$$
$$\Longrightarrow p(t) = c_1 t + c_2 \text{ and } h(x) = c_3.$$

Putting in equation (4.31), gives

$$A = \frac{1}{4a+1}[c_1 t + c_2 - c_3], \tag{4.33}$$

Putting equation (4.33) in equation (4.26), we get

$$\frac{-4a}{4a+1}[c_1 t + c_2 - c_3] = c_3, \tag{4.34}$$

Comparing coefficients in equation (4.34)

$$\frac{-4a}{4a+1}c_1 = 0, \tag{4.35}$$

Since $a \neq 0$, we have

$$\Longrightarrow c_1 = 0,$$
$$\Longrightarrow A = \frac{1}{4a+1}[c_2 - c_3], \tag{4.36}$$
$$\Longrightarrow A = k, \quad \text{(where } k \text{ is a suitable constant)} \tag{4.37}$$

So, equation (4.6) becomes

$$\tau_t = 0. \tag{4.38}$$

So, we have

$$\phi(x, y, t, u) = ku + B(x, y, t), \tag{4.39}$$

where $B(x, y, t)$ satisfies the original PDE.

Also

If $\tau_t = 0$, then $\xi_x = 0$.

and $\tau_{xx} = 0$, implies that $\xi_{tt} = 0$.

Now

$\tau_t = 0$ and $\tau_u = 0$ implies that $\tau = \tau(x, y)$,

and $\tau_{xx} = 0$ implies that τ is linear in x,

$$\Longrightarrow \tau = f(y)x + g(y), \tag{4.40}$$
$$\Longrightarrow \tau_x = f(y). \tag{4.41}$$

Put in equation (e_{10}),

$$\cot x f(y) + \csc^2 x \left(f^{''}(y)x + g^{''}(y)\right) = 0,$$
$$\Longrightarrow \cot x f(y) + x\csc^2 x f^{''}(y) + \csc^2 x g^{''}(y) = 0, \tag{4.42}$$

This is possible only if

$$f(y) = 0 \Longrightarrow \tau_x = 0, \tag{4.43}$$

and $g^{''}(y) = 0$.

So

$$\tau = \tau(y) = g(y) \text{ with } g^{''}(y) = 0 \Longrightarrow g(y) = \alpha y + \beta,$$

$$\Longrightarrow \tau = \alpha y + \beta. \tag{4.44}$$

From $(e_8) \Longrightarrow \xi_t = 0$,

$$(e_3)_t = 0 \Longrightarrow \theta_{xt} = 0, \tag{4.45}$$

So, equation (3.11) becomes

$$-2\cot x \tau_y = 0 \Longrightarrow \tau_y = 0, \tag{4.46}$$

which implies that

$$\tau = c, \qquad \text{(where } c \text{ is some constant).} \tag{4.47}$$

Now

$$\xi_x = \xi_t = \xi_u = 0 \text{ implies that}$$

$$\xi = \xi(y). \tag{4.48}$$

From equation (e_{12}), we get

$$\csc^2 x \xi_{yy} + \csc^2 x \xi = 0 \Longrightarrow \xi_{yy} + \xi = 0,$$

$$\Longrightarrow \xi = k_1 \cos y + k_2 \sin y. \tag{4.49}$$

Now, from equation (e_7), we get

$$\csc^2 x \tau_y - \theta_t = 0 \Longrightarrow \theta_t = 0. \tag{4.50}$$

So

$$\theta_t = \theta_u = 0, \text{ implies that}$$

$$\theta = \theta(x, y). \tag{4.51}$$

From equation (e_3), we get

$$\Longrightarrow \theta_x = -\csc^2 x\left[-k_1 \sin y + k_2 \cos y\right], \tag{4.52}$$

and from equation (e_6), we get

$$\Longrightarrow \theta_y = -\cot x\left[k_1 \cos y + k_2 \sin y\right], \tag{4.53}$$

Integrating equation (4.52) with respect to x, we get

$$\theta(x, y) = \cot x\left[-k_1 \sin y + k_2 \cos y\right] + v(y), \tag{4.54}$$

Using equation (4.54) in equation (4.53), we get

$$-\cot x\left[k_1 \cos y + k_2 \sin y\right] + v^{'}(y) = -\cot x\left[k_1 \cos y + k_2 \sin y\right],$$

$$\Longrightarrow v^{'}(y) = 0 \Longrightarrow v(y) = k_4,$$

$$\Longrightarrow \theta(x, y) = \cot x\left[-k_1 \sin y + k_2 \cos y\right] + k_4. \tag{4.55}$$

So, we have

$$\xi = k_1 \cos y + k_2 \sin y,$$

$$\tau = k_3,$$

$$\theta = \cot x\left[-k_1 \sin y + k_2 \cos y\right] + k_4,$$

$$\phi = k_5 u + B\left(x, y, t\right),$$

where $B\left(x, y, t\right)$ satisfies the original PDE.

Now

Putting $k_1 = 1$ and the remaining constants vanish implies

$$X_1 = \cos y \frac{\partial}{\partial x} - \sin y \cot x \frac{\partial}{\partial y}. \tag{4.56}$$

Putting $k_2 = 1$ and the remaining constants vanish implies

$$X_2 = \sin y \frac{\partial}{\partial x} + \cos y \cot x \frac{\partial}{\partial y}. \tag{4.57}$$

Putting $k_3 = 1$ and the remaining constants vanish implies

$$X_3 = \frac{\partial}{\partial t}. \tag{4.58}$$

Putting $k_4 = 1$ and the remaining constants vanish implies

$$X_4 = \frac{\partial}{\partial y}. \tag{4.59}$$

Putting $k_5 = 1$ and the remaining constants vanish implies

$$X_5 = u \frac{\partial}{\partial u}. \tag{4.60}$$

So: the equation

$$u_{tt} = u_{xx} + \left(\cot x\right) u_x + \left(\csc^2 x\right) u_{yy} + au,$$

admits 5-dimensional Lie algebra generated by

$$X_1 = \cos y \frac{\partial}{\partial x} - \sin y \cot x \frac{\partial}{\partial y},$$

$$X_2 = \sin y \frac{\partial}{\partial x} + \cos y \cot x \frac{\partial}{\partial y},$$

$$X_3 = \frac{\partial}{\partial t},$$

$$X_4 = \frac{\partial}{\partial y},$$

$$X_5 = u \frac{\partial}{\partial u},$$

with commutation relations given in the table below

Table 4

Commutator table for the Lie algebra

	X_1	X_2	X_3	X_4	X_5
X_1	0	$-X_4$	0	X_2	0
X_2	X_4	0	0	$-X_1$	0
X_3	0	0	0	0	0
X_4	$-X_2$	X_1	0	0	0
X_5	0	0	0	0	0

4.2 Symmetry reductions and invariant solutions

We are going to reduce the following equation

$$u_{tt} = u_{xx} + (\cot x)\, u_x + \left(\csc^2 x\right) u_{yy} + au. \tag{4.61}$$

4.2.1 Reduction Using Subalgebra $\langle X_3, X_4 \rangle = \left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial y} \right\rangle$

Starting with the first symmetry which can be written as

$$X = 0\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} + \frac{\partial}{\partial t} + 0\frac{\partial}{\partial u}, \tag{4.62}$$

The characteristic system

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{0},$$

From $\frac{dx}{0} = \frac{dt}{1}$, we have

$$x = c,$$

and from $\frac{dy}{0} = \frac{dt}{1}$, we get

$$y = c,$$

and from $\frac{dt}{1} = \frac{du}{0}$, we get

$$u = c,$$

So, we get three invariants

$$q\,(x, y, t) = x, \tag{4.63}$$

$$v\,(x, y, t) = y, \tag{4.64}$$

$$P\,(q, v) = u, \tag{4.65}$$

Now:

$$u_t = P_q q_t + P_v v_t = 0, \tag{4.66}$$

$$u_{tt} = 0, \tag{4.67}$$

$$u_x = P_q q_x + P_v v_x = P_q, \tag{4.68}$$

$$u_{xx} = P_{qq}, \tag{4.69}$$

$$u_y = P_q q_y + P_v v_y = P_v, \tag{4.70}$$

$$u_{yy} = P_{vv}, \tag{4.71}$$

Substituting in equation (4.61), we get

$$P_{qq} + (\cot q)\, P_q + \left(\csc^2 q\right) P_{vv} + aP = 0. \tag{4.72}$$

The second symmetry X_4 is inherited by PDE (4.61), as it commutes with X_3.

So, using the second symmetry $\overline{X} = X_4 = \frac{\partial}{\partial y}$,

$$X = \overline{X}\,(q)\,\frac{\partial}{\partial q} + \overline{X}\,(v)\,\frac{\partial}{\partial v} + \overline{X}\,(P)\,\frac{\partial}{\partial P},$$

$$\overline{X}\,(q) = \overline{X}\,(x) = 0,$$

$$\overline{X}\,(v) = \overline{X}\,(y) = 1,$$

$$\overline{X}\,(P) = \overline{X}\,(u) = 0,$$

So, the second symmetry can be written as

$$X = 0\tfrac{\partial}{\partial q} + \tfrac{\partial}{\partial v} + 0\tfrac{\partial}{\partial P}, \tag{4.73}$$

The characteristic system

$$\tfrac{dq}{0} = \tfrac{dv}{1} = \tfrac{dP}{0},$$

From $\frac{dq}{0} = \frac{dv}{1}$, we have

$$q = c,$$

and from $\frac{dv}{1} = \frac{dP}{0}$, we get

$$P = c,$$

So, the invariants are

$$\xi\,(q, v) = q, \tag{4.74}$$

$$\omega\,(\xi) = P, \tag{4.75}$$

Now:

$$P_v = \omega_\xi \xi_v = 0, \tag{4.76}$$

$$P_{vv} = 0, \tag{4.77}$$

$$P_q = \omega_\xi \xi_q = \omega^{'}, \tag{4.78}$$

$$P_{qq} = \omega^{''}, \tag{4.79}$$

Substituting in equation (4.72), we get the ODE

$$\omega^{''} + \cot\xi\omega^{'} + a\omega = 0, \tag{4.80}$$

Equation (4.80) has the symmetry

$$X = \cos\xi\tfrac{\partial}{\partial\omega}, \tag{4.81}$$

We find the first prolongation of X,

$$\eta^{[1]} = \eta_\xi + \left(\eta_\omega - \psi_\xi\right)\omega^{'} - \psi_\omega\omega^{'^2},$$

$$\eta^{[1]} = -\sin\xi, \tag{4.82}$$

So

$$X^{[1]} = 0\tfrac{\partial}{\partial\xi} + \cos\xi\tfrac{\partial}{\partial\omega} - \sin\xi\tfrac{\partial}{\partial\omega'}, \tag{4.83}$$

The characteristic system

$$\tfrac{d\xi}{0} = \tfrac{d\omega}{\cos\xi} = \tfrac{d\omega^{'}}{-\sin\xi},$$

From $\frac{d\xi}{0} = \frac{d\omega}{\cos\xi}$, we have

$$\xi = c, \tag{4.84}$$

and from $\frac{d\omega}{\cos\xi} = \frac{d\omega^{'}}{-\sin\xi}$, we get

$$\cos\xi d\omega^{'} = -\sin\xi d\omega,$$

Integrating both sides using equation (4.84), we get

$$\cos\xi\omega' = -\sin\xi\omega + c,$$

$$\Longrightarrow \cos\xi\omega' + \sin\xi\omega = c, \tag{4.85}$$

So, the invariants are

$$r = \xi, \tag{4.86}$$

$$Y = \cos\xi\omega' + \sin\xi\omega, \tag{4.87}$$

So, equation (4.80) is reduced to the first order ODE

$$\tfrac{dY}{dr} = -\cot rY, \tag{4.88}$$

which implies that

$$\tfrac{dY}{Y} = -\cot r dr \Longrightarrow \ln Y = \ln(\sin r) + c,$$

$$\Longrightarrow Y(r) = \tfrac{c}{\sin r}, \tag{4.89}$$

Substitute the similarity variables in equation (4.89)

$$\cos\xi\omega' + \sin\xi\omega = \tfrac{c}{\sin\xi},$$

$$\Longrightarrow \omega' + \tan\xi\omega = \tfrac{c}{\sin\xi\cos\xi}, \tag{4.90}$$

The integrating factor of equation (4.90) is

$$e^{\int} \tan\xi d\xi = e^{\ln(\sec\xi)} = \sec\xi,$$

So, equation (4.90) becomes

$$\sec\xi\omega' + \sec\xi\tan\xi\omega = \tfrac{c}{\sin\xi\cos^2\xi}, \tag{4.91}$$

$$\Longrightarrow \tfrac{d}{d\xi}(\sec\xi\omega) = \tfrac{c}{\sin\xi\cos^2\xi}, \tag{4.92}$$

$$\Longrightarrow \sec\xi\omega = \tfrac{c}{\sin\xi\cos^2\xi}d\xi,$$

$$\Longrightarrow \omega = \tfrac{c}{\sin\xi\cos\xi}d\xi, \tag{4.93}$$

Therefore, equation (4.93) has the solution

$$\omega(\xi) = C_1 + C_2\cos\xi\ln(\csc\xi - \cot\xi), \tag{4.94}$$

Substituting back, we get

$$P = C_1 + C_2\cos q\ln(\csc q - \cot q), \tag{4.95}$$

which implies that

$$u(x,y,t) = C_1 + C_2\cos x\ln(\csc x - \cot x). \tag{4.96}$$

is a solution of equation (4.61).

4.2.2 Reduction Using Subalgebra $\langle X_3, X_2 \rangle = \left\langle \frac{\partial}{\partial t}, \sin y \frac{\partial}{\partial x} + \cos y \cot x \frac{\partial}{\partial y} \right\rangle$

Starting with the first symmetry which can be written as

$$X = 0\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} + \frac{\partial}{\partial t} + 0\frac{\partial}{\partial u}, \tag{4.97}$$

The characteristic system

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{0},$$

From $\frac{dx}{0} = \frac{dt}{1}$, we have

$$x = c,$$

and from $\frac{dy}{0} = \frac{dt}{1}$, we get

$$y = c,$$

and from $\frac{dt}{1} = \frac{du}{0}$, we get

$$u = c,$$

So, we get three invariants

$$q(x, y, t) = x, \tag{4.98}$$

$$v(x, y, t) = y, \tag{4.99}$$

$$P(q, v) = u, \tag{4.100}$$

Now:

$$u_t = P_q q_t + P_v v_t = 0, \tag{4.101}$$

$$u_{tt} = 0, \tag{4.102}$$

$$u_x = P_q q_x + P_v v_x = P_q, \tag{4.103}$$

$$u_{xx} = P_{qq}, \tag{4.104}$$

$$u_y = P_q q_y + P_v v_y = P_v, \tag{4.105}$$

$$u_{yy} = P_{vv}, \tag{4.106}$$

So, equation (4.61) becomes

$$P_{vv} = P_{qq} + (\cot q) P_q + \left(\csc^2 q\right) P_{vv} + aP. \tag{4.107}$$

The second symmetry X_2 is inherited by PDE (4.61), as it commutes with X_3.

So, using the second symmetry $\overline{X} = X_2 = \sin y \frac{\partial}{\partial x} + \cos y \cot x \frac{\partial}{\partial y}$,

$$X = \overline{X}(q) \frac{\partial}{\partial q} + \overline{X}(v) \frac{\partial}{\partial v} + \overline{X}(P) \frac{\partial}{\partial P},$$

$$\overline{X}(q) = \overline{X}(x) = \sin v,$$

$$\overline{X}(v) = \overline{X}(y) = \cos v \cot q,$$

$$\overline{X}(P) = \overline{X}(u) = 0,$$

So, the second symmetry can be written as

$$X = \sin v \frac{\partial}{\partial q} + \cos v \cot q \frac{\partial}{\partial v} + 0\frac{\partial}{\partial P}, \tag{4.108}$$

The characteristic system

$$\frac{dq}{\sin v} = \frac{dv}{\cos v \cot q} = \frac{dP}{0},$$

From $\frac{dq}{\sin v} = \frac{dv}{\cos v \cot q}$, we have

$$\cos v \cot q dq = \sin v dv \Longrightarrow \cot q dq = \tan v dv,$$

$$\Longrightarrow \ln(\sin q) = -\ln(\cos v) + c \Longrightarrow \ln(\sin q \cos v) = c,$$

$$\Longrightarrow \sin q \cos v = c,$$

and from $\frac{dq}{\sin v} = \frac{dP}{0}$, we get

$$P = c,$$

So, the invariants are

$$\xi(q, v) = \sin q \cos v, \tag{4.109}$$

$$\omega(\xi) = P, \tag{4.110}$$

Now:

$$P_q = \omega_\xi \xi_q = \cos q \cos v \omega', \tag{4.111}$$

$$P_{qq} = \cos^2 q \cos^2 v \omega'' - \sin q \cos v \omega', \tag{4.112}$$

$$P_v = \omega_\xi \xi_v = -\sin q \sin v \omega', \tag{4.113}$$

$$P_{vv} = \sin^2 q \sin^2 v \omega'' - \sin q \cos v \omega', \tag{4.114}$$

Substituting in equation (4.107), we get $\left(\cos^2 q \cos^2 v \omega'' - \sin q \cos v \omega'\right) + \cot q \left(\cos q \cos v \omega'\right) + \csc^2 q \left(\sin^2 q \sin^2 v \omega'' - \sin q \cos v \omega'\right) + f(\omega) = 0,$

$\Longrightarrow \left(\cos^2 q \cos^2 v + \csc^2 q \sin^2 q \sin^2 v\right) \omega'' - \left(\sin q \cos v - \cot q \cos q \cos v + \csc^2 q \sin q \cos v\right) \omega' + f(\omega) = 0,$

$$\Longrightarrow \left(1 - \xi^2\right) \omega'' - 2\xi \omega' + a\omega = 0, \tag{4.115}$$

which for $a = n(n+1)$, is Legendre's Equation of order n, leading to solution of equation (4.61) in terms of Legendre Functions.

4.2.3 Reduction Using Subalgebra $\langle \alpha X_3 + \beta X_4, cX_3 + \lambda X_5 + eX_4 \rangle = \left\langle \alpha \frac{\partial}{\partial t} + \beta \frac{\partial}{\partial y}, c\frac{\partial}{\partial t} + \lambda u \frac{\partial}{\partial u} + e\frac{\partial}{\partial y} \right\rangle$

Starting with the first symmetry which can be written as

$$X = 0\frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} + \alpha \frac{\partial}{\partial t} + 0\frac{\partial}{\partial u}, \tag{4.116}$$

The characteristic system

$$\frac{dx}{0} = \frac{dy}{\beta} = \frac{dt}{\alpha} = \frac{du}{0},$$

From $\frac{dx}{0} = \frac{dt}{1}$, we have

$$x = c,$$

and from $\frac{dy}{\beta} = \frac{dt}{\alpha}$, we get

$$\alpha y - \beta t = c,$$

and from $\frac{dt}{\alpha} = \frac{du}{0}$, we get

$$u = c,$$

So, we get three invariants

$$q\left(x,y,t\right) = x, \tag{4.117}$$

$$v\left(x,y,t\right) = \alpha y - \beta t, \tag{4.118}$$

$$P\left(q,v\right) = u, \tag{4.119}$$

Now:

$$u_t = P_q q_t + P_v v_t = -\beta P_v, \tag{4.120}$$

$$u_{tt} = \beta^2 P_{vv}, \tag{4.121}$$

$$u_x = P_q q_x + P_v v_x = P_q, \tag{4.122}$$

$$u_{xx} = P_{qq}, \tag{4.123}$$

$$u_y = P_q q_y + P_v v_y = \alpha P_v, \tag{4.124}$$

$$u_{yy} = \alpha^2 P_{vv}, \tag{4.125}$$

So, equation (4.61) becomes

$$\beta^2 P_{vv} = P_{qq} + \cot q P_q + \alpha^2 \csc^2 q P_{vv} + aP. \tag{4.126}$$

The second symmetry X_2 is inherited by PDE (4.61), as it commutes with X_3.

So, using the second symmetry $\overline{X} = cX_3 + \lambda X_5 + eX_4 = 0\frac{\partial}{\partial x} + e\frac{\partial}{\partial y} + c\frac{\partial}{\partial t} + \lambda u\frac{\partial}{\partial u}$,

$$X = \overline{X}\left(q\right)\frac{\partial}{\partial q} + \overline{X}\left(v\right)\frac{\partial}{\partial v} + \overline{X}\left(P\right)\frac{\partial}{\partial P},$$

$$\overline{X}\left(q\right) = \overline{X}\left(x\right) = 0,$$

$$\overline{X}\left(v\right) = \overline{X}\left(\alpha y - \beta t\right) = e\alpha - c\beta,$$

$$\overline{X}\left(P\right) = \overline{X}\left(u\right) = \lambda u = \lambda P,$$

So, the second symmetry can be written as

$$X = 0\frac{\partial}{\partial q} + \left(e\alpha - c\beta\right)\frac{\partial}{\partial v} + \lambda P\frac{\partial}{\partial P}, \tag{4.127}$$

The characteristic system

$$\frac{dq}{0} = \frac{dv}{e\alpha - c\beta} = \frac{dP}{\lambda P},$$

From $\frac{dq}{0} = \frac{dv}{e\alpha - c\beta}$, we have

$$q = c,$$

and from $\frac{dv}{e\alpha - c\beta} = \frac{dP}{\lambda P}$, we get

$$\ln\left(P\right) - \frac{\lambda}{e\alpha - c\beta}v = c,$$

So, the invariants are

$$\xi\left(q,v\right) = q, \tag{4.128}$$

$$\omega\left(\xi\right) = \ln\left(P\right) - \frac{\lambda}{e\alpha - c\beta}v, \tag{4.129}$$

Equation (4.129) implies

$$\Longrightarrow \ln(P) = \omega + \frac{\lambda}{e\alpha - c\beta}v,$$

$$\Longrightarrow P = e^{\omega}e^{\frac{\lambda}{e\alpha - c\beta}v}, \tag{4.130}$$

Now

$$P_q = e^{\omega}e^{\frac{\lambda}{e\alpha - c\beta}v}\omega^{'} = P\omega^{'}, \tag{4.131}$$

$$P_{qq} = P\omega^{'^2} + P\omega^{''}, \tag{4.132}$$

$$P_v = \frac{\lambda}{e\alpha - c\beta}e^{\omega}e^{\frac{\lambda}{e\alpha - c\beta}v} = \frac{\lambda}{e\alpha - c\beta}P, \tag{4.133}$$

$$P_{vv} = \frac{\lambda^2}{(e\alpha - c\beta)^2}P, \tag{4.134}$$

Substituting in equation (4.126), we get

$$\frac{\beta^2\lambda^2}{(e\alpha - c\beta)^2}P = P\omega^{'^2} + P\omega^{''} + \cot\xi P\omega^{'} + \frac{\alpha^2\lambda^2}{(e\alpha - c\beta)^2}\csc^2\xi P + aP,$$

$$\Longrightarrow \frac{\beta^2\lambda^2}{(e\alpha - c\beta)^2} = \omega^{'^2} + \omega^{''} + \cot\xi\omega^{'} + \frac{\alpha^2\lambda^2}{(e\alpha - c\beta)^2}\csc^2\xi + a, \tag{4.135}$$

where $\alpha \neq 0, \beta \neq 0, \lambda \neq 0$,

For $a = 1, e = 0, \lambda = c$, equation (4.135) becomes

$$\omega^{''} + \omega^{'^2} + \cot\xi\omega^{'} + \frac{\alpha^2}{\beta^2}\csc^2\xi = 0, \tag{4.136}$$

Let

$$\omega^{'} = Y, \tag{4.137}$$

Equation (4.136) becomes

$$Y^{'} + Y^2 + \cot\xi Y + \frac{\alpha^2}{\beta^2}\csc^2\xi = 0, \tag{4.138}$$

which has the solution

$$Y(\xi) = \frac{\alpha}{\beta}\tan\left[C_1 + \frac{\alpha}{\beta}\ln\left(\frac{1+\cos\xi}{\sin\xi}\right)\right]\csc\xi, \tag{4.139}$$

For $C_1 = 0$ and $\alpha = \beta$, equation (4.139) becomes

$$\omega^{'} = \tan\left[\ln\left(\frac{1+\cos\xi}{\sin\xi}\right)\right]\csc\xi, \tag{4.140}$$

Let

$$r = \ln\left(\frac{1+\cos\xi}{\sin\xi}\right), \tag{4.141}$$

$$\Longrightarrow dr = -\csc\xi d\xi, \tag{4.142}$$

So, equation equation (4.140) becomes

$$\omega^{'} = -\tan r dr, \tag{4.143}$$

integrating equation (4.143), we get

$$\omega(\xi) = C + \ln(\cos r),$$

$$\Longrightarrow \omega(\xi) = C + \ln\left[\cos\left(\ln\left(\frac{1+\cos\xi}{\sin\xi}\right)\right)\right], \tag{4.144}$$

Substituting back, we get

$$\ln(P) = -\frac{v}{\beta} + C + \ln\left[\cos\left(\ln\left(\frac{1+\cos\xi}{\sin\xi}\right)\right)\right],$$

$$\Longrightarrow P = Ce^{-\frac{v}{\beta}}\cos\left(\ln\left(\frac{1+\cos\xi}{\sin\xi}\right)\right), \tag{4.145}$$

which implies that

$$u(x,y,t) = Ce^{-y+t}\cos\left(\ln\left(\tfrac{1+\cos x}{\sin x}\right)\right). \tag{4.146}$$

is a solution of equation (4.61).

Conclusion

In conclusion, I would like to summarize the main results of this thesis. In the first chapter, I used basic differential geometry to formulate the Klein Gordon equations on a sphere.

Then in chapter two, I briefly described the Lie symmetry method for ODE's and PDE's in three independent variables.

In chapter three, I considered the nonlinear Klein Gordon equation on a sphere. I found the minimal symmetry algebra of the equation and proved that the equation admits four symmetries for any nonlinear function $f(u)$. Then, I determined all forms of $f(u)$ which may give more symmetry algebra, and proved that all these functions will give only the minimal symmetry algebra. Then, I used two dimensional subalgebras to reduce the equation into ODE and obtained some exact solutions using the special function $f(u) = -(e^u + 2)$.

Finally in chapter four, I considered the linear Klein Gordon equation on a sphere. I found the symmetry algebra of the equation and proved that the equation admits five symmetries. Then, I used two dimensional subalgebras to reduce the equation into ODE and obtained some exact solutions.

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